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REAL HYPERSURFACES IN COMPLEX CENTRO-AFFINE SPACES

V. Cruceanu

One of the important geometries, subordinated to the affine one and tight related to it, is the centro-affine geometry. The bases have been built by G.Tzitzeica and a decisive role in its development had O.Mayer. Important contributions have been also brought by A.Myller, I.Popa, E.Salkowski, P.Delens, V.Vagner, D.Laugwitz, E.Calabi, a.a.

By attempting a study of the submanifolds in a complex affine space, we get to the conclusion that it is necessary to start with an analogous research in the centro-affine (c.a.) one. This is the reason for dealing in this work with the study of an important class of real hypersurfaces in the complex centro-affine space (c.a.s.).

1. Let C_n be an n-dimensional real c.a.s. and let O be its center. By considering the associated vector space V_n, endowed with the canonical c.a. structure, the mapping $f : C_n \to V_n$ which associates to each point P of C_n its position vector $r = f(P)$ of V_n, establishes an isomorphism between C_n and V_n as c.a.s. and enables to transfer the vectorial structure from V_n to C_n. By identifying V_n with C_n by the mapping f we can regard C_n either as a pointwise c.a.s. or as a vector-space. By considering then a c.a. frame in C_n, the equation of a hyperplane α ,

which does not pass by the center O, can be written in the form

(1) $$\alpha_p x^p = 1, \quad (p=1,2,\ldots,n).$$

To the change of the frame, (α_p) transform as the coordinates of a covector ρ on V_n. We denote by C_n^* the set of non-central hyperplanes in C_n and associate to each hyperplane α of C_n^* the covector $\rho \in V_n^*$ given by the equation (1). Thus we obtain a bijection which is independent of the choice of the frame, and permits to transfer the canonical c.a. structure and the vectorial one from V_n^* to C_n^*. With this c.a. structure, C_n^* is called the dual of C_n. It follows that on C_n is valid the principle of the projective duality which confers to its geometry a special beauty.

To facilitate the exposition we shall use in the following, with some exception, the notations and the terminology from [5], extended to the n-dimensional case. Since the space C_n has an absolut parallelism, we could identify a tangent vector at a point P in C_n with its equipolent by O.

2. Let S be a hypersurface in the real c.a.s. C_n given locally by

(2) $$r=r(u^i) \quad \text{or dually} \quad \rho = \rho(u^i), \qquad (i,j,k=1,2,\ldots,n),$$

r and ρ being C^∞-functions and satisfying the conditions

(3) $$(r,r_1,\ldots,r_{n-1})\neq 0, \qquad (\rho, \rho_1,\ldots, \rho_{n-1})\neq 0,$$

where $r_i = \partial r/\partial u^i$ and $\rho_i = \partial\rho/\partial u^i$.

From the condition of incidence (1) we have

(4) $$\rho(u^i)r(u^i)=1,$$

and, ρ being the covector of the tangent hyperplane, we obtain

(5) $$\rho r_i = \rho_i r = 0$$

and then

(6) $$\rho\, \partial_j r_i = -\rho_j r_i = \partial_j \rho_i r.$$

By taking into account relations (3) we obtain for the hypersurface S the fundamental equations

(7) $$\partial_j r_i = \Gamma^k_{ji} r_k + G_{ji} r \quad \text{and dually} \quad \partial_j \rho_i = {}'\Gamma^k_{ji} \rho_k + {}'G_{ji}\rho$$

One can see that Γ and $'\Gamma$ are symmetrical linear connections and $G='G$ is a (0,2)-symmetrical nondegenerate tensor field on S. The integrability conditions for (7_1) and (7_2) respectively, are

(8) $$R^h_{kji} = G_{ki}\delta^h_j - G_{ji}\delta^h_k \,, \qquad \nabla_k G_{ji} = \nabla_j G_{ki} \,,$$

(9) $$'R^h_{kji} = G_{ki}\delta^h_j - G_{ji}\delta^h_k \,, \qquad '\nabla_k G_{ji} = '\nabla_j G_{ki} \,,$$

which show that Γ and $'\Gamma$ are equiprojective connections and ∇G and $'\nabla G$ are completely symmetric tensors. The two connections are related by the condition

(10) $$\partial_k G_{ij} - \Gamma^h_{ki} G_{hj} - '\Gamma^h_{kj} G_{ih} = 0 \,,$$

which expresses that they are conjugate with respect to G.

By considering the mean connection and the tensor of deformation for the two connections Γ and $'\Gamma$,

(11) $$\bar{\Gamma}^k_{ji} = \frac{1}{2}(\Gamma^k_{ji} + '\Gamma^k_{ji}), \qquad h^k_{ji} = \frac{1}{2}(\Gamma^k_{ji} - '\Gamma^k_{ji})$$

and putting

(12) $$h_{ijk} = h^h_{ij} G_{hk}$$

one obtains

(13) $$\bar{\nabla}_k G_{ji} = 0 \,, \qquad h_{ijk} = h_{ikj} \,.$$

Hence the mean connection is the Levi-Civita connection of G and h_{ijk} is a completely symmetric tensor field. We have also

(14) $$\nabla_k G_{ji} = -'\nabla_k G_{ji} = -2h_{kji}, \qquad \bar{\nabla}_k h^h_{ji} = \bar{\nabla}_j h^h_{ki} \,.$$

Putting

(15) $$t_j = \frac{1}{n-1} h^i_{ji}$$

we get from (14_3)

(16) $$\bar{\nabla}_k t_j = \bar{\nabla}_j t_k$$

that is, the 1-form $t=t_j du^j$ is closed. We shall prove that this form is just exact [3]. First, we take a frame in C_n and consider the euclidean metric in which this frame becomes orthonormed. Then denoting by K the total curvature of the hypersurface S and by d the distance from the center O to the tangent hyperplane one finds that

(17) $$T = Kd^{-n-1}$$

is a relative c.a. invariant of the hypersurface S and one obtains for t the expression

(18) $$t = -\frac{1}{2(n-1)}\, d\ln|T|\,.$$

The invariant T was discovered by Tzitzeica [6] in 1907 for the case n=3. The surfaces T=constant (or t=0) were introduced and studied by Tzitzeica too. They were called by Jonas, Demulin, Loria and others the Tzitzeica surfaces and by Blaschke and his school, the affine spheres of center O. The invariant T and the surfaces T=constant have been extended for arbitrary n by Tzitzeica himself and others,[2].

3. Let now a complex c.a.s. of complex dimension n. We shall regard it as a real c.a.s. C_{2n} of real dimension 2n endowed with an automorphism J satisfying

(19) $$J^2 = -I\,.$$

On the c.a.s. C^*_{2n} of noncentral hyperplanes of C_{2n} we obtain the complex structure defined by $J'=J^{-1}=-J$.

Among the hypersurfaces S of C_{2n} we distinguish those for which the vector of the c.a. normal at any point of S belongs to the tangent hyperplane at this point and which we shall call special hypersurfaces.

One raises the following questions: if there exist such hypersurfaces, how general are they and which are their geometric properties.

To answer these questions, we consider in C_{2n} a c.a. frame, adapted to the complex structure J and the euclidean metric for which it is orthonormed. Assuming the hypersurface S given locally by the equation

(20) $$F(x^1,\dots,x^n,x^{n+1},\dots,x^{2n})=0,$$

it will be special iff the vector J(r) will be orthogonal to the vector grad F. Since $r=(x^p,x^{n+p})$, $J(r)=(-x^{n+p},x^p)$ and grad $F = (F_p,F_{n+p})$ with $F_p=\partial F/\partial x^p$, $(p=1,2,\dots,n)$, we must have

(21) $$-x^{n+p}F_p + x^pF_{n+p} = 0 .$$

The characteristic system of this equation is

(22) $$\frac{dx^p}{dt} = -x^{n+p} , \quad \frac{dx^{n+p}}{dt} = x^p , \quad (p=1,2,..,n)$$

and it has the general solution

(23) $$x^p=A^p\cos t-B^p\sin t, \quad x^{n+p}=A^p\sin t+B^p\cos t, \quad t\in R,$$

with A^p and B^p arbitrary constants. By putting $E=(A^p,B^p)$ the solution can be written in the form

(24) $$r(t) = E\cos t + J(E)\sin t , \quad t\in R,$$

and thus the characteristic curves of equation (22) are the circles centered at the point O, situated in holomorphic central planes. So we have

<u>Proposition 1.</u> A hypersurface S in C_{2n} is special iff it is generated by arcs of holomorphic central circles which pass by the points of a codimension 2 submanifold in C_{2n}, that is transverse at every point of it to the holomorphic central cercle which passes by that point.

Supposing such a submanifold given by the local equation

(25) $$r = r_1(u^1,\ldots,u^{2n-2}),$$

we obtain for the corresponding special hypersurface

(26) $$r = r_1(u^1,\ldots,u^{2n-2})\cos t + J(r_1(u^1,\ldots,u^{2n-2}))\sin t.$$

By observing that for the characteristic system (22) we have the following 2n-1 independent prime integrals,

$$(x^p)^2+(x^{n+p})^2=C^p,\ p=1,2,..,n, \qquad x^1x^{n+p'}-x^{n+1}x^{p'}=D^{p'},\ p'=2,3,..n,$$

a special hypersurface can be given locally by an equation of the form

$$F((x^1)^2+(x^{n+1})^2,\ldots,(x^n)^2+(x^{2n})^2,x^1x^{n+2}-x^{n+1}x^2,\ldots,x^1x^{2n}-x^{n+1}x^n)=0,$$

where F is a C^∞-function, which satisfies some regularity conditions resulted from (3). Consequently, the class of special hypersurfaces is sufficiently large.

For such a hypersurface we can write

(29) $J(r)=\xi^{i}r_{i}$, $J(r_{j})=\varphi_{j}^{i}r_{i}-\eta_{j}r$,

where ξ , φ , η are respectively a vector field, a (1,1)-tensor field and a 1-form on S. By applying once again the automorphism J we get

(30) $\eta_{i}\xi^{i}=1$, $\varphi_{j}^{i}\xi^{j}=0$, $\eta_{i}\varphi_{j}^{i}=0$, $\varphi_{k}^{i}\varphi_{j}^{k}=-\delta_{j}^{i}+\eta_{j}\xi^{i}$

that is

Proposition 2. On a special hypersurface S, the complex structure J of the space C_{2n} induces an almost-contact structure (ξ, φ, η).

Adapting the notion of CR-submanifold, introduced by A. Bejancu in hermitian manifold [1], to the case of hypersurfaces in c.a.s., we shall call proper CR-hypersurface in the complex c.a.s. (C_{2n},J) a hypersurface S with the property that there exists on it two suplementary distributions Δ and $\bar{\Delta}$ such that

$$J(\Delta_{P})=\Delta_{P}, \quad J(\bar{\Delta}_{P})=\{\lambda r_{P}, \lambda\in R\}, \quad \forall P\in S.$$

It is easy to see now that it holds

Proposition 3. A necessary and sufficient condition that a hypersurface S in (C_{2n},J) be special is to be a proper CR-hypersurface.

From the relation

$$\partial_{j}J(r)=J(r_{j}), \quad \partial_{j}J(r_{i})=J(\partial_{j}r_{i})$$

we obtain, taking into account (30) and (7),

(31) $\eta_{j}=-G_{ji}\xi^{i}$, $\nabla_{j}\xi^{i}=\varphi_{j}^{i}$, $\nabla_{j}\eta_{i}=\Phi_{ji}$, $\nabla_{j}\varphi_{i}^{k}=G_{ji}\xi^{k}+\delta_{j}^{k}\eta_{i}$,

where we have put

(32) $$\Phi_{ji}=G_{jk}\varphi_{i}^{k}.$$

Denoting by L_{ξ} the Lie derivative with respect to ξ , we deduce from (3) and (31),

(33) $L_{\xi}\xi=0$, $L_{\xi}\eta=0$, $L_{\xi}\varphi=0$, $L_{\xi}G=0$

and hence,

Proposition 4. The one parametric group generated by the vector field ξ preserves the structure (ξ, φ, η, G) on a special hypersurface S.

We also have

$$\nabla_\xi \xi = 0, \quad \nabla_\xi \eta = 0, \quad \nabla_\xi \varphi = 0, \tag{34}$$

that is

Proposition 5. The trajectories of the field ξ are autoparallel curves for the connection ∇, the hyperplanes of the distribution $\eta_i du^i = 0$ are parallel and the endomorphism defined by φ preserves the parallel transport along these trajectories.

From (30) and (31) we obtain

$$N^i_{jk} + (\nabla_j \eta_k - \nabla_k \eta_j)\xi^i = 0, \tag{35}$$

where N is the Nijenhuis tensor for φ, that is we have

Proposition 6. The almost contact structure (ξ, φ, η), induced by the complex structure J of the space C_{2n}, on a special hypersurface S, is normal.

By duality, putting

$$J'(\rho) = {}'\xi^i \rho_i + {}'\xi^o \rho, \qquad J'(\rho_j) = {}'\varphi^i_j \rho_i - {}'\eta_j \rho$$

and taking into account (4)-(7) and (29) we get

$${}'\xi^o = 0, \quad {}'\xi^i = \xi^i, \quad {}'\eta_j = \eta_j, \quad G_{ki}\,{}'\varphi^i_j + G_{ji}\,\varphi^i_k = 0. \tag{36}$$

So, the notion of special hypersurface is autodual. By multiplying the last relation (36) with φ^k_h and by taking into account (30) we find

$$G_{ki}\,\varphi^k_h\,{}'\varphi^i_j = G_{hj} + \eta_h \eta_j. \tag{37}$$

By supressing the accents for ξ and η we can write the formulas

$$J'(\rho) = \xi^i \rho_i, \qquad J'(\rho_j) = {}'\varphi^i_j \rho_i - \eta_j \rho. \tag{38}$$

Applying once again the automorphism J' we get

$$\eta_i \xi^i = 1, \quad {}'\varphi^i_j \xi^j = 0, \quad \eta_i\,{}'\varphi^i_j = 0, \quad {}'\varphi^i_k\,{}'\varphi^k_j = -\delta^i_j + \eta_j \xi^i, \tag{39}$$

i.e., the structure $(\xi, {}'\varphi, \eta)$ is of almost-contact too. From

$$\partial_j J'(\rho) = J'(\rho_j), \qquad \partial_j J'(\rho_i) = J(\partial_j \rho_i)$$

and the fundamental equations (7_2), it follows

$$(40)\qquad \eta_j = -G_{ji}\xi^i,\quad {}'\nabla_j\xi^i = {}'\varphi^i_j,\quad {}'\nabla_j\eta_i = {}'\Phi_{ji},$$
$$ {}'\nabla_j{}'\varphi^k_i = G_{ji}\xi^k + \delta^k_j\eta_i,$$

where we have put

$$(41)\qquad {}'\Phi_{ji} = G_{jk}{}'\varphi^k_i .$$

We obtain then

$$(42)\qquad {}'N^i_{jk} + ({}'\nabla_j\eta_k - {}'\nabla_k\eta_j)\xi^i = 0,$$

and thus the almost-contact structure $(\xi, {}'\varphi, \eta)$ is also normal.

There are two problems which appear: when the two almost-contact structures coincide and when one of them is pseudo-sasakian with respect to tensor G. As an answer firstly we have

Proposition 7. For a special hypersurface the following 21 conditions are equivalent:

a) $h^k_{ij}\xi^j = 0$, b) $\eta_k h^k_{ij}=0$, c) ${}'\varphi^i_j=\varphi^i_j$, d) $\nabla_j\xi^i={}'\nabla_j\xi^i$,

e) $\nabla_j\eta_i={}'\nabla_j\eta_i$, f) $\nabla_k G_{ij}\xi^k=0$, g) ${}'\nabla_k G_{ij}\xi^k=0$,

h) $\nabla_j\eta_i-\nabla_i\eta_j=\Phi_{ji}$, i) ${}'\nabla_j\eta_i-{}'\nabla_i\eta_j={}'\Phi_{ji}$,

j) $G_{hk}\varphi^h_i\varphi^k_j=G_{ij}+\eta_i\eta_j$, k) $G_{hk}{}'\varphi^h_i{}'\varphi^k_j=G_{ij}+\eta_i\eta_j$,

l) $\nabla_k\varphi^i_j={}'\nabla_k\varphi^i_j$, m) $\nabla_k{}'\varphi^i_j={}'\nabla_k{}'\varphi^i_j$, n) $h^k_{ij}\varphi^j_h=\varphi^k_j h^j_{ih}$,

o) $h^k_{ij}{}'\varphi^j_h={}'\varphi^k_j h^j_{ih}$, p) $\Phi_{ij}=-\Phi_{ji}$, q) ${}'\Phi_{ij}=-{}'\Phi_{ji}$,

r) $\bar\nabla_k\varphi^i_j=G_{kj}\xi^i+\delta^i_k\eta_j$, s) $\bar\nabla_k{}'\varphi^i_j=G_{kj}\xi^i+\delta^i_k\eta_j$,

t) the structure (ξ,φ,η,G) is pseudo-sasakian,

u) the structure $(\xi,{}'\varphi,\eta,G)$ is pseudo-sasakian.

One shows rather easily that each of the previous conditions is equivalent to the condition a). We shall only prove that j) implies a) because the argument is here more complicated.

From (12),(31),(37),(39), we find

(43) $$G_{hk}\varphi^h_i\varphi^k_j - G_{ij} - \eta_i\eta_j = 2\varphi^k_i h_{jkh}\xi^h$$

and therefore j) implies $\varphi^k_i h_{jkh}\xi^h = 0$. Since rank φ=2n-1 and h is symmetric, it follows that there exists a function λ so that

(44) $$h_{ijh}\xi^h = \lambda\eta_i\eta_j \ .$$

Differentiating covariantly with respect to the mean connection, we obtain

$$\bar\nabla_k h_{ijh}\xi^h + h_{ijh}\varphi^h_k = \lambda_k\eta_i\eta_j + \lambda(\bar\nabla_k\eta_i\eta_j + \bar\nabla_i\eta_k\eta_j + \eta_k h_{ijh}\xi^h).$$

Contracting by φ^j_m and using (44) we find

$$\bar\nabla_k h_{ijh}\xi^h\varphi^j_m + h_{ijh}\varphi^j_m\varphi^h_k = \lambda\eta_i\bar\nabla_k\eta_j\varphi^j_m \ .$$

Changing i with k and substracting, it results

$$h_{ijh}\varphi^j_m\varphi^h_k - h_{kjh}\varphi^j_m\varphi^h_i = \lambda(\eta_i\bar\nabla_k\eta_j - \eta_k\bar\nabla_i\eta_j)\varphi^j_m \ .$$

Contracting by ξ^i and using (44) and j), we get

$$\lambda(G_{km} - \eta_k\eta_m) = 0 \ .$$

Because G is nondegenerate, it follows $\lambda = 0$ and from (44) it results a). Next we have

<u>Proposition 8.</u> On a special hypersurface S in the complex c.a.s., the almost-contact structure (ξ, φ, η, G) (or $(\xi, '\varphi, \eta, G)$) is pseudo-sasakian iff S is a piece of a nondegenerate special hypercuadric having the point O as center.

Indeed, taking in the condition a) from Proposition 7, i=k and summing, we obtain

(45) $$t_j\xi^j = 0.$$

Then putting i=k in condition n) and summing it follows

$$h^i_{ij}\varphi^j_h = \varphi^k_j h^j_{kh} = \Phi_{kj}h^{kj}_h = 0,$$

because Φ is skew-symmetric and h is symmetric. We have thus $\varphi^j_i t_j = 0$ and so there exists a function μ such that $t_j = \mu\eta_j$. Contracting here with ξ^j and taking into account (45) it results $\mu = 0$ and $t_j = 0$. That is S is a Tzitzeica hypersurface of center O.

Now, differentiating condition a) in the mean connection and using (31),(4),(14_2) and (45) we obtain

$$\bar{\nabla}_h h^k_{ij}\,\xi^j = -h^k_{ij}\,\varphi^j_h = -h^j_{ih}\,\varphi^k_j = \bar{\nabla}_j h^j_{ih}\xi^k = \bar{\nabla}_i h^j_{jh}\xi^k = (2n-1)\,\bar{\nabla}_i t_h \xi^k = 0$$

From here, it follows $h^k_{ij}\varphi^j_h = 0$ and thus there are the functions ν^k_i so that $h^k_{ij} = \nu^k_i \eta_j$. Contracting here with ξ^j and using a) we obtain $\nu^k_i = 0$ and thus

(46) $$h^k_{ij} = 0 \ ,$$

i.e., S is a piece of a hypercuadric of center O [5]. Conversely, for a nondegenerate hypercuadric of center O holds relation (46) and therefore condition a) is verified. Hence, from Proposition 7, the structure (ξ, φ, η, G) on it is pseudo-sasakian.

Let now S be a nondegenerate hypercuadric of center O. Its equation can be written in the form

(47) $$A_{\alpha\beta}\, x^\alpha x^\beta = 1, \qquad (\alpha, \beta, \gamma = 1,2,\dots,2n),$$

with $\det(A_{\alpha\beta}) \neq 0$. The tangent hyperplan of S at a point $P_o(r_o)$ of it has the equation

$$A_{\alpha\beta}\, x^\alpha x^\beta_o = 1 \ .$$

Hence the vector $J(r_o)$ satisfies the condition (29_1) iff

$$A_{\alpha\beta}\, J^\alpha_\gamma\, x^\gamma_o x^\beta_o = 0 \ .$$

This relation is equivalent to

(48) $$A_{\alpha\beta}\, J^\alpha_\gamma J^\beta_\delta = A_{\gamma\delta}$$

which, at its turn, is equivalent with the fact that every point P(r) of the hypercuadric is mapped by J in a point P'(J(r)) belonging also to hypercuadric. We have thus

<u>Proposition 9.</u> A nondegenerate hypercuadric of center O is special iff it is invariant relative to the automorphism J, which defines the complex structure on C_{2n}.

Condition (48) is also equivalent with the fact that the matrix A in adapted frames has the expression

(49) $$A = \begin{bmatrix} a_{pq} & b_{pq} \\ -b_{pq} & a_{pq} \end{bmatrix}, \quad (p,q=1,2,\ldots,n),$$

where $a_{pq} = a_{qp}$ and $b_{pq} = -b_{qp}$. Putting $\alpha_{pq} = a_{pq} + ib_{pq}$ and $z^p = x^p + ix^{n+p}$, the equation (47) can be written in the form

(50) $$\alpha_{pq} z^p \bar{z}^q = 1$$

and thus we have

Proposition 10. The nondegenerate special hypercuadrics of center O are given by equation (50) where (α_{pq}) is a nonsingular hermitian matrix.

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Universitatea "AL.1.Cuza"
Seminarul Matematic "A.Myller"
6600 IASI, R.S. Romania