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A SURVEY ON PARACOMPLEX GEOMETRY

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1. Introduction. We shall call paracomplex geometry the geometry related to the algebra of paracomplex numbers [**70**] and, mainly, the study of the structures on differentiable manifolds called paracomplex structures. When, moreover, we consider a compatible neutral pseudo-Riemannian metric, we have the para-Hermitian and para-Kähler structures, and their variants.

This subject has been studied, since the first papers by Rashevskij [**94**], Libermann [**69**] and Patterson [**90**] until now, from several different points of view. Moreover, the papers related to it have appeared many times in a rather disperse way, and the different schools or authors have worked many times having no relation with one another. So, the interest for a survey on the topic is clear; and, of course, mathematicians, and in particular differential geometers, will find a lot of definitions, examples, results and references in paracomplex geometry. Furthermore, we think that the solved problems in paracomplex geometry are only a little part of the "first" questions to be studied in the theory. And this is one of the attractive features of it: to be *de facto* a young branch of differential geometry, with many significant results to be proved, as the reading of the present survey, we hope, shows. Nowadays, the subject has already applications to several topics, as: negatively curved manifolds and Anosov diffeomorphisms [**47**], quantizable coadjoint orbits [**57**], mechanics [**11, 86**], chronogeometry [**82**], elliptic geometry [**32**], pseudo-Riemannian space forms [**38**], etc.

On the other hand, as is seen in certain recent works, paracomplex geometry is related to some physical problems, and we think that this work can also be useful for physicists and mathematicians working in them. In particular, since every almost para-Kählerian manifold is symplectic, those manifolds furnish a large family of symplectic manifolds.

Moreover, as Kaneyuki [**49**] points out, the study of global geometric

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properties of affine symmetric spaces seems to be an interesting and important problem, since affine symmetric spaces have begun to play a central role in the theory of group representations, as is shown in [**13, 30, 87**]. In para-Hermitian geometry, a type of affine symmetric spaces, called para-Hermitian symmetric spaces, arises (see Section 5). These spaces are useful, among other things, in the representation theory of the associated groups. Moreover, those symmetric spaces M are diffeomorphic to (a covering manifold of) the cotangent bundle of another—Riemannian—symmetric space M_0. That permits us to bring into $T^*(M_0)$ the para-Kähler structure on M, thus obtaining an additional structure on the cotangent bundle of those manifolds, which is of great interest in symplectic geometry and mechanics.

Paracomplex geometry is a topic with many analogies and also with differences with complex geometry. Thus this subject is often studied with the geometries arising from complex numbers, and also with the geometry coming from dual numbers, see for instance [**21, 26, 70, 109, 118**]. However, for the sake of brevity, we shall make, in this survey, no references to related results on other algebras, such as complex numbers or dual numbers. On the other hand, a little knowledge of paracomplex geometry shows soon that there are more differences with complex geometry than one can suspect at the beginning.

Perhaps, the subject would we worthy to appear in the corresponding section of the A.M.S. Subject Classification. We have entitled the present work "A survey on paracomplex geometry," but it is perhaps premature to definitively name the topic since there are several different names, each of which have a reasonable sense, and mainly because the growth of the subject can help to fix the most reasonable name.

2. Paracomplex manifolds.

2.1. Some definitions and results. We shall first recall some general definitions concerning (almost) paracomplex, (almost) para-Hermitian and (almost) para-Kähler manifolds. From now on, all the manifolds and geometric objects are supposed to be C^∞.

Definition 2.1. An *almost product structure* J on a differentiable manifold M is a $(1,1)$ tensor field J on M such that $J^2 = 1$. The pair (M, J) is called an *almost product manifold*. An *almost paracomplex* *manifold* is an almost product manifold (M, J) such that the two eigenbundles T^+M and T^-M associated with the two eigenvalues $+1$ and -1 of J, respectively, have the same rank. (Note that the dimension of an almost paracomplex manifold is necessarily even.) Equivalently, a splitting of the tangent bundle TM of a differentiable manifold M, into the Whitney sum of two subbundles $T^{\pm}M$ of the same fiber dimension is called an *almost paracomplex structure on* M. An almost paracomplex structure on a $2n$-dimensional manifold M may alternatively be defined as a G-structure on M with structural group $\mathrm{GL}\,(n, \mathbf{R}) \times \mathrm{GL}\,(n, \mathbf{R})$.

A *paracomplex manifold* is an almost paracomplex manifold (M, J) such that the G-structure defined by the tensor field J is integrable. An integrable almost product manifold is usually called a *locally product manifold*. Thus, a paracomplex manifold is a locally product manifold (M, J) such that if the characteristic polynomial of J is $(x-1)^r(x+1)^s$, $r+s = \dim M$, then $r = s$. We can give another—equivalent—definition of paracomplex manifold in terms of local homeomorphisms with the space $\mathbf{A}^n$ [**70**] and paraholomorphic changes of charts, in a way similar to the complex case.

Definition 2.2. Let (M, J) and (M', J') be (almost) paracomplex manifolds. Then a smooth map f of M to M' is called a *paraholomorphic map* if the relation $f_{*p} \circ J_p = J'_{f(p)} \circ f_{*p}$ is satisfied for each point $p \in M$, where f_{*p} is the differential of f at p. If there is a paraholomorphic diffeomorphism of M onto M', then (M, J) and (M', J') are said to be paraholomorphically equivalent. A paraholomorphic diffeomorphism of M onto itself is called a paraholomorphic transformation of M. We denote by $\mathrm{Aut}\,(M, J)$ the group of paraholomorphic transformations of M.

2.2. Examples of paracomplex manifolds. (1) The product manifold $M^n \times M^n$ of a real manifold by itself has a canonical paracomplex structure.

(2) [**55, 70**]. Let $(x_1, \dots, x_n, y_1, \dots, y_n)$ be the natural coordinates on $\mathbf{R}^{2n}$. Let us consider the following two kinds of foliations: $x_k + y_k =$ const, and $x_k - y_k =$ const, $1 \leq k \leq n$, which define a paracomplex structure on $\mathbf{R}^{2n}$. These foliations are invariant under translations by the lattice $\mathbf{Z}^{2n}$ of all integral points in $\mathbf{R}^{2n}$. So, they naturally induce

a paracomplex structure on the torus $\mathbf{R}^{2n}/\mathbf{Z}^{2n}$.

(3) [**82**]. The group $G = \mathrm{SL}(2,\mathbf{R})$ acts on its Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{R})$ by conjugation. As the invariant bilinear form $(X,Y) = \mathrm{Tr}\,XY/2$ on $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{R})$ has signature (2,1), Ad defines a double covering of $\mathrm{SL}(2,\mathbf{R})$ onto $\mathrm{SO}_0(2,1)$. Choose the basis

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{R})$, and use the corresponding coordinates $(x,y,z) \leftrightarrow xX + yY + zZ$ to identify $\mathfrak{g}$ with $\mathbf{R}^3$. The G-orbits in $\mathfrak{g}$ are of several types. The *hyperbolic orbits*, defined by $G\cdot(\lambda X) = G\cdot(\lambda Y)$, $\lambda > 0$, are diffeomorphic to the hyperbola $G\cdot(\lambda X) \approx Q_{+\lambda} = \{(x,y,z) : x^2+y^2-z^2 = \lambda^2\}$. These orbits are non-Riemannian symmetric spaces, where the corresponding involution σ is given by conjugation by X:

$$\sigma\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix},$$

and the corresponding fixpoint group is

$$H = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbf{R} - \{0\}\right\}.$$

The tangent space of M at $(\lambda,0,0) = \lambda X$ is now identified with $T_{(\lambda,0,0)}Q_{+\lambda} = \mathbf{R}Y \oplus \mathbf{R}Z = \mathfrak{m}$, and the involution $(a,b) \mapsto (b,a)$ defined by $(1/2)\mathrm{ad}_{\mathfrak{m}}X$ is a paracomplex structure commuting with $\mathrm{Ad}(H)$ making $Q_{+\lambda}$ a paracomplex manifold.

(4) Many authors, among whom are C. Bejan, V. Cruceanu, S. Ianuş, S. Ishihara, T. Nagano, V. Oproiu, R. Rosca, C. Udriste and K. Yano, have considered almost paracomplex structures on the tangent bundle of a manifold M. Let ∇ be a linear connection on M, and denote by X^v and X^h the vertical and horizontal lift, respectively, to the tangent bundle TM [**122**] of the vector field $X \in \mathfrak{X}(M)$. Putting then

(2.1)
$$P(X^v) = X^v, \quad P(X^h) = -X^h, \quad Q(X^v) = X^h, \quad Q(X^h) = X^v,$$

they obtain two almost paracomplex structures on TM. The structure P is paracomplex if and only if ∇ has vanishing curvature, and Q is paracomplex if and only if ∇ has both vanishing torsion and curvature. These structures have been extended to the case of a nonlinear connection, and to the specific cases of a nonlinear connection defined by a Finsler, Lagrange or Hamilton structure [**11, 86**]. Similar structures for the cotangent bundle are obtained from a connection ∇ and a nondegenerate (0,2) tensor field g on M [**25**]. If α is a differentiable 1-form and X a vector field on M, α^v denotes the vertical lift of α and X^h the horizontal lift of X to T^*M, putting

$$P(X^h) = -X^h, \quad P(\alpha^v) = \alpha^v, \quad Q(X^h) = (X^\flat)^v, \quad Q(\alpha^v) = (\alpha^\sharp)^h, \tag{2.2}$$

where $\flat$ and $\sharp$ are the g-musical isomorphisms, they obtain two almost paracomplex structures on T^*M. P is paracomplex if ∇ has vanishing curvature and Q is paracomplex if and only if both the exterior covariant differential Dg of g given by

$$(Dg)(X, Y) = \nabla_X(Y^\flat) - \nabla_Y(X^\flat) - [X, Y]^\flat, \qquad X, Y \in \mathfrak{X}(M),$$

and the curvature of ∇ vanish. The case in which ∇ is symmetric has been considered in [**9**]. In this reference one can find more examples.

2.3. Historical remarks on neutral and paracomplex geometries. We now give some historical remarks on paracomplex geometry, including para-Kähler manifolds. We do not consider the most recent results, which are included in the different sections of the present survey, according to the specific topic.

Graves [**41**] introduced in 1845 a generalization of complex numbers, by considering expressions of the type $xi + yj$, where x and y are real numbers, and the symbols i and j satisfy certain relations. That construction includes complex numbers (if $i = 1$ and $j^2 = -1$), and also paracomplex numbers (if $i = 1$ and $j^2 = 1$). Graves applies his general construction to solve some questions in number theory, but he does not give any specific application of paracomplex numbers.

Clifford wrote, about 1873, four papers [**17, 18, 19, 20**], where he began the study of the compatibility of the non-Euclidean geometries with mechanics by using, among other tools, Hamilton's quaternions. He mainly considered elliptic geometry and found the general expression of a transformation which sends a so-called *motor*—whose simplest instance is the instantaneous "rotation + translation" of a rigid body—in another motor. He proved that such a transformation is of the type $q_1 + q_2 j$, where q_1 and q_2 are both quaternions, and j is the so-called *rectangular twist of unit pitch*, or, more specifically, the equivalence class of all such twists; a twist is a quantity having direction, sense and module. He also proved that in the three-dimensional elliptic space one has $j^2 = 1$. Here we find again an "involutive" operator, related in this case to a couple of quaternions, and an expression of the type $x + yj$. Nevertheless, we cannot yet speak of paracomplex numbers in the present sense.

Kotelnikov [**65**] in 1895 and Study [**114**] in 1903 used paracomplex numbers as a tool for a more comfortable computation in the study of certain phenomena in Mechanics and in submanifolds of Euclidean space. As Rozenfeld [**105**] points out, what motivated Clifford's, Kotelnikov's and Study's investigations was the desire to know whether or not the geometry of non-Euclidean space contradicted the principles of mechanics. Kotelnikov and Study developed a theory of sliding vectors in these spaces. In the space $\mathbf{R}^3$, a system of forces and a system of instantaneous angular velocities—a force and a instantaneous angular velocity are both sliding vectors—are equivalent, respectively, to a screw force and a kinematic twist. The first of them consists of a force and a couple of forces in a plane perpendicular to it, and the second of the angular velocity of rotation around some axis and the translation velocity along this axis, which can be regarded as a couple of angular velocities. Kotelnikov and Study proved also that in the elliptic and Lobatchevskian spaces, every system of sliding vectors is equivalent to two sliding vectors whose lines of action are two reciprocal polars.

Octav Mayer [**74, 75, 76**] introduced in 1938 the concept of *hyperbolic biaxial geometry*. In the spirit of Felix Klein's Program of Erlangen [**62**], Mayer considered as space the projective space $P_3(\mathbf{R})$ and as group the collineations which preserve two arbitrary fixed nonincident lines. The given projective space is obtained from the four dimensional space $\mathbf{R}^4$, and the two invariant lines arise from two supplementary planes of $\mathbf{R}^4$. If we consider these planes as the eigenspaces of an almost product structure operator we obtain a paracomplex structure on $\mathbf{R}^4$. The projective group which preserves the two initial fixed lines in $P_3(\mathbf{R})$ corresponds to the subgroup of $\mathrm{GL}(4,\mathbf{R})$ which preserves the two corresponding planes in $\mathbf{R}^4$, or, equivalently, to the subgroup of matrices which commute with the almost product operator.

Mayer's ideas were followed in two directions. The first one, by considering general dimensions or general reductible groups. In this line, D. Papuc [**89**] considers the submanifolds of a projective space which have as automorphism group a completely reductible projective subgroup. A particular case is that of Mayer's biaxial space.

The second one, by extending the algebraic and synthetical aspects. In this sense, Rozenfeld [**106**] defined explicitly paracomplex geometry. Both algebraic and synthetical approaches are also considered in [**118**].

Rozenfeld's book [**109**] is a veritable treatise on paracomplex geometry (and other subjects). The author studies, in a very detailed way, paracomplex affine, Euclidean, pseudo-Euclidean, para-Kähler (called unitarian), projective and conformal spaces, giving real models for each of these spaces. This book is an important work on the algebraic and synthetical aspects of spaces over algebras. A translation would be welcome.

Rashevskij [**94**] introduced in 1948 the properties of para-Kähler manifolds, when he considered a metric of signature (n, n) defined from a potential function, the so-called *scalar field*, on a $2n$-dimensional locally product manifold—called by him *stratified space*. Para-Kähler manifolds were explicitly defined by Rozenfeld [**106**] in 1949. Rozenfeld compared there Rashevskij's definition with Kähler's definition in the complex case and established the analogy between complex and paracomplex ones.

Then, Rozenfeld [**106**] remarked that Rashevskij's spaces are the real model of para-Kähler manifolds. So, *Rozenfeld was the first author who consciously worked on paracomplex geometry*, using Graves-Clifford's double numbers. Moreover, Rozenfeld introduced in that paper the space of 0-couples P_n^0 as a model of projective paracomplex space. In [**108**], Rozenfeld considers the projective space (denoted $P_n(e)$ by him) over the "double," i.e., paracomplex, numbers $\mathbf{A}$, and gives a one-to-one correspondence between it and P_n^0. He also studies the analytical m-surfaces; actually, he considers mainly hypersurfaces, in those spaces. In [**1, 2**], among other subjects, Rozenfeld's investigations are continued with the study of submanifolds of codimension greater than 1 in the space of 0-couples P_n^0. In [**113**], a comparative study is made between Kähler and para-Kähler structures using complex and paracomplex numbers. The author also gives some historical remarks.

Independently, paracomplex manifolds, almost para-Hermitian manifolds and para-Kähler manifolds were defined by Libermann [**69**] in 1952 and [**70**] in 1954, in the context of G-structures and with other geometrical structures. This work contains a lot of definitions and theorems, and constitutes a main reference on paracomplex geometry. We adopt in this survey some definitions and terminology introduced by Libermann in it.

Para-Kähler spaces were independently defined by Ruse [**111**], who considered in detail the case of nonflat pseudo-Riemannian spaces that admit two non-intersecting null parallel 2-planes, Patterson [**90, 91**] and Wong [**120**], who worked in the context of parallel distributions (see Theorem 4.1), and seeking canonical forms of metrics, especially with respect to symmetric and harmonic cases. As Olszak [**85**] points out, these spaces were called Kähler spaces by Patterson (and Wong) because the defining condition was formally similar to the usual Kähler condition.

The book by Vyshnevkij, Shirokov and Shurygin [**118**] is a monography in which the authors study differential geometry on manifolds over general algebras. In particular they study the paracomplex case, especially the unitary paracomplex; that is, para-Kähler, manifolds. The book contains also an extensive bibliography on the subject of manifolds over algebras. A translation would also be welcome.

We quote here only one more paper: Prvanović's paper [**93**], which deserves a special mention, as it not only contains several interesting definitions and results on paracomplex and para-Hermitian geometries, but encouraged the research on the subject, since one can see many differences between complex and paracomplex geometry. Prvanović introduced, among other things, the paraholomorphic projective curvature tensor, and also gave the explicit expression of the curvature tensor for spaces with constant paraholomorphic sectional curvature.

Finally, we mention only a concept linked with paracomplex geometry: the *reflectors*, which were introduced in the study of neutral surfaces in four-dimensional neutral pseudo-Riemannian manifolds [**46**] and are the neutral space analogs of the twistor spaces of Riemannian geometry.

3. Para-Hermitian manifolds.

3.1. Almost para-Hermitian manifolds.

Definition 3.1. An *almost para-Hermitian manifold* (M, g, J) is a differentiable manifold M endowed with an almost product structure J and a pseudo-Riemannian metric g, compatible in the sense that

$$g(JX, Y) + g(X, JY) = 0, \qquad X, Y \in \mathfrak{X}(M). \tag{3.1}$$

An *almost para-Hermitian structure* on a differentiable manifold M is a G-structure on M whose structural group is the real representation of the paraunitary group $\mathrm{U}(n, \mathbf{A})$. An almost para-Hermitian manifold can also be defined as a differentiable manifold with an almost para-Hermitian structure. It is easy to check that an almost para-Hermitian manifold is necessarily almost paracomplex and that the metric g has signature (n, n).

A *para-Hermitian manifold* is a manifold with an integrable almost para-Hermitian structure (g, J). That is, the G-structure associated with J is integrable [**33**].

We shall call the 2-covariant skew-symmetric tensor field F defined by $F(X, Y) = g(X, JY)$, the *fundamental* 2*-form* of the almost para-Hermitian manifold (M, g, J).

An *almost para-Kähler manifold* is an almost para-Hermitian manifold (M, g, J) such that $dF = 0$.

We say that two almost para-Hermitian manifolds (M, g, J) and (M', g', J') are *paraholomorphically isometric* if there exists an isometry $f\colon M \to M'$ such that $f_* \circ J = J' \circ f_*$.

For the definition of paraholomorphic sectional curvature and other definitions related to it, see [**93, 34**].

3.2. Examples of almost para-Hermitian manifolds. (1) [**10**]. The cotangent bundle of any manifold admits an almost para-Hermitian structure. One way to give such a structure is the following: Let M be an n-dimensional differentiable manifold, endowed with a symmetric linear connection ∇ with coefficients Γ^k_{ij}. Then, one can define the Riemann extension on the cotangent bundle T^*M as the pseudo-Riemannian metric G on the total space of T^*M locally given by

$$G = \begin{pmatrix} -2p_k\Gamma_{ij}^k & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix},$$

with regard to the local coordinates (x_i, p_i) on T^*M. On the other hand, let J be the almost product structure on T^*M which has as vertical distribution the vertical subbundle of TT^*M, and as horizontal subbundle the horizontal distribution determined by the connection ∇. We have the following result:

Theorem 3.2 [**10**]. *With the above notations, (G, J) is an almost para-Kähler structure on the total space of the cotangent bundle T^*M, whose fundamental 2-form F satisfies $F = d\theta$ (where θ denotes the Liouville form), and thus coincides with the canonical symplectic structure on T^*M. Moreover, if ∇ has vanishing curvature, then the structure (G, J) is para-Kähler.*

(2) [**10**]. Let $S_n^{2n-1}(r) = \{x \in \mathbf{R}_n^{2n} : \langle x, x\rangle = r^2\}$ be the pseudosphere of radius $r \geq 0$, dimension $2n-1$ and index n in $\mathbf{R}_n^{2n}$, and let $H_n^{2n-1}(r) = \{x \in \mathbf{R}_{n+1}^{2n} : \langle x, x\rangle = -r^2\}$ be the pseudohyperbolic space of radius $r \geq 0$, dimension $2n-1$ and index n in $\mathbf{R}_{n+1}^{2n}$. Then the product manifolds $M = S_{n_1}^{2n_1-1}(r_1) \times S_{n_2}^{2n_2-1}(r_2)$ and $M = H_{n_1}^{2n_1-1}(r_1) \times H_{n_2}^{2n_2-1}(r_2)$, $r_1, r_2 \geq 0$, $n_1, n_2 \in \{1, 2, \dots\}$, admit a family of almost para-Hermitian structures. By analogy with Hopf's and Calabi-Eckmann's manifolds, Bejan calls *hyperbolic Hopf manifolds* and *hyperbolic Calabi-Eckmann manifolds* to the above product manifold M when either $n_1 = 1$ and $n_2 \in \{2, 3, \dots\}$ or $n_1, n_2 \in \{2, 3, \dots\}$, respectively.

(3) A lot of examples of almost para-Hermitian manifolds are given by Bejan in [**10**], some of which are included in the present survey. Those given in [**9**] exhibit almost para-Hermitian structures on the tangent bundle TM of a manifold M, associated with vertical, complete and horizontal lifts [**122**] of tensor fields on M to TM.

(4) Ianuş and Rosca [**45**] proved that, given a Riemannian structure g on a manifold M, and the almost paracomplex structure P on TM given in (2.2), being ∇ the Levi-Civita connection of g, then the complete lift g^c of g with respect to ∇ determines an almost para-Hermitian structure on TM.

(5) [**70**]. The pseudosphere S^6_3 admits a structure of almost para-Hermitian manifold which is not integrable. Libermann obtains this structure by using Cayley's split octaves.

3.3. Representation-theoretical classification of almost para-Hermitian manifolds. As is well-known, in [**42**] almost Hermitian manifolds (M, g, J) are classified with respect to the decomposition in invariant and irreducible subspaces, under the action of the structural group $\mathrm{U}(n)$, of the vector space of tensors satisfying the same symmetries as the covariant derivative ∇F of the fundamental 2-form F with respect to the Levi-Civita connection ∇ of the metric g. Thus, we have an adequate framework for the several types of almost Hermitian manifolds, previously defined by a number of authors in terms of geometric properties which retain some portion of Kähler geometry. A classification of almost para-Hermitian manifolds is made in [**8**]. The author obtains 36 classes up to duality, and gives characterizations of some of the classes. A classification à la Gray-Hervella is given in [**37**], where 136 classes up to duality are obtained. We give here the table of primitive classes $\mathcal{W}_1, \dots, \mathcal{W}_8$ obtained in [**37**]:

TABLE 1. Primitive classes of almost para-Hermitian manifolds of dimension ≥ 6.

	$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_3$	$\mathcal{W}_4$	$\mathcal{W}_5$	$\mathcal{W}_6$	$\mathcal{W}_7$	$\mathcal{W}_8$
$(\nabla_X F)(X,Y) = 0$	*				*			
$dF = 0$		*				*		
$\delta F = 0$			*				*	
$(\nabla_X F)(Y,Z) =$ $\{1/2(n-1)\}\{\delta F(Y)g(X,Z)$ $-\delta F(Z)g(X,Y)$ $+\delta F(JY)g(X,JZ)$ $-\delta F(JZ)g(X,JY)\}$				*				*
$\nabla_A B \in \mathcal{V}$			*	*	*	*	*	*
$\nabla_U B \in \mathcal{V}$	*	*	*	*	*	*		
$\nabla_A U \in \mathcal{H}$	*	*			*	*	*	*
$\nabla_U V \in \mathcal{H}$	*	*	*	*			*	*

Here, for an almost para-Hermitian manifold (M, g, J), ∇ denotes the

Levi-Civita connection, F the fundamental 2-form, δ the codifferential, $\mathcal{V}$ (for *vertical*) the $(+1)$-eigendistribution associated with the eigenvalue $+1$ of J, $\mathcal{H}$ (for *horizontal*) the (-1)-eigendistribution corresponding to the eigenvalue -1 of J, A, B vector fields of $\mathcal{V}$ and U, V vector fields of $\mathcal{H}$.

Remark 3.3. Locally conformal para-Kähler manifolds (which belong to the class $\mathcal{W}_4 \oplus \mathcal{W}_8$) are the paracomplex analogs of locally conformal Kähler manifolds, introduced in [**117**]. They have been studied in [**37**], where an analog of the Weyl conformal tensor is introduced.

Example 3.4 [**10**]. The product manifold $M = S^1 \times H^{2n-1}_{n-1}(r)$, $r > 0$, $n \in \{2, 3, \dots\}$, with the structure given in that reference, is a locally conformal para-Kähler manifold which cannot be globally conformal para-Kähler.

3.4. Homogeneous almost para-Hermitian structures. The classical characterization by Ambrose and Singer [**5**] of homogeneous Riemannian manifolds in terms of a (1,2) tensor field S on the manifold, which is an extension of Cartan's characterization [**15**] of Riemannian symmetric manifolds (for which one has $S = 0$) is extended to pseudo-Riemannian manifolds of arbitrary signature in [**39**]. The authors give the following:

Definition 3.5. A *homogeneous almost para-Hermitian structure* on the almost para-Hermitian manifold (M, g, J) is a (1,2) tensor field S on M such that the connection $\tilde{\nabla} = \nabla - S$, where ∇ denotes the Levi-Civita connection of the metric g, parallelizes the metric g, its curvature R and the tensor fields J and S.

Theorem 3.6 [**39**]. *Let (M, g, J) be a connected, simply connected and complete almost para-Hermitian manifold of dimension $2n$. Then the following conditions are equivalent*:

(a) *(M, g, J) admits a homogeneous almost para-Hermitian structure.*

(b) *(M, g, J) is a reductive homogeneous almost para-Hermitian manifold $(M = G/H, g, J)$.*

(c) *The restriction of the Killing-Cartan form of* $\mathfrak{so}(n,n)(T_pM)$ *to the Lie algebra* $\mathfrak{h}$ *of the isotropy group* H *via the representation* ρ *obtained from the Kostant operator* (*see* [**39**]) *is nondegenerate.*

Notice that in the Riemannian case a homogeneous manifold is always complete and reductive.

Example 3.7 [**39**]. Libermann's quadric [**70**] S^6_3 can be viewed as the pseudo-Riemannian manifold of constant curvature $O(3,4)/O(3,3)$, which is pseudo-Riemannian symmetric, with corresponding homogeneous pseudo-Riemannian structure $S = 0$. But it can also be considered as the homogeneous space $G'_2/\mathrm{SL}(3,\mathbf{R})$, where G'_2 denotes the exceptional simple Lie group which is the second real form of the complex group of which the usual group G_2 is the compact real form. The isotropy group is the special paraunitary group isomorphic to the real special linear group of order 3. The homogeneous space $G'_2/\mathrm{SL}(3,\mathbf{R})$ admits an almost para-Hermitian structure (g,J) which is not para-Kähler, but nearly para-Kähler [**10**], and it is a reductive almost para-Hermitian manifold with homogeneous almost para-Hermitian structure $S = -(1/2)J \circ (\nabla J)$, where ∇ denotes the Levi-Civita connection of g.

3.5. Examples of para-Hermitian manifolds. (1) [**22**]. A type of para-Hermitian structures arises as a particular case of the general situation given by the tangent bundle TM of a manifold M endowed with a linear connection and a tensor field of type (1,1) or (0,2).

On the other hand, the behavior of para-Hermitian structures on a manifold with regard to the action of the multiplicative group of nonsingular tensor fields of type (1,1) on the tensor algebra, on its algebra of derivations and on the affine module of linear connections is studied as a particular case in [**24**].

(2) [**46**]. The two-dimensional case has some interesting features: Let (M,g) be an oriented two-dimensional neutral manifold. The metric g and the orientation of M induce a unique almost para-Hermitian structure J, which is automatically integrable since M is two-dimensional, and it can thus be proved that there exist a kind of "*isothermal coordinates*" x, y, such that the metric can be locally

written as $g = 2\rho\, dx\, dy$, where ρ is a positive C^∞ function.

(3) [**46**]. Let $G_{1,1}(2,2) \approx \mathrm{SO}\,(2,2)/\mathrm{SO}\,(1,1) \times \mathrm{SO}\,(1,1)$ be the Grassmannian of oriented neutral planes in $\mathbf{R}^4_2$. Then $G_{1,1}(2,2)$ admits a para-Hermitian structure obtained from an $\mathrm{SO}\,(2,2)$-invariant metric related to the Maurer-Cartan form of $\mathrm{SO}\,(2,2)$.

4. Para-Kähler manifolds.

4.1. Definitions and first properties.

Definition 4.1. A para-Hermitian manifold (M, g, J) is said to be a *para-Kähler manifold* if $dF = 0$. Equivalently [**23, 55**], a *para-Kähler manifold* is an almost para-Hermitian manifold such that $\nabla J = 0$, where ∇ denotes the Levi-Civita connection of g. We can also define, from [**85**] or from the classification in Subsection 3.3, a *para-Kähler manifold* as a pseudo-Riemannian manifold of dimension $2n$ endowed with two n-dimensional totally isotropic and parallel distributions $\mathcal{V}$ and $\mathcal{H}$ such $\mathcal{V} \cap \mathcal{H} = \{0\}$.

Given a connected (almost) para-Hermitian or para-Kähler manifold (M, g, J) and denoting by $I(M, g)$ the isometry group of M with respect to g, the *automorphism group* of (M, g, J) is defined as

$$\mathrm{Aut}\,(M, g, J) = \mathrm{Aut}\,(M, g) \cap \mathrm{Aut}\,(M, J);$$

it is a closed subgroup of $\mathrm{Aut}\,(M, g)$, and consequently a Lie transformation group of M. If $\mathrm{Aut}\,(M, g, J)$ acts transitively on M, then (M, g, J) is called a *homogeneous* (*almost*) *para-Hermitian* or *para-Kähler manifold*, according to whether it is (almost) para-Hermitian or para-Kähler. Notice that a homogeneous almost para-Kähler manifold is a homogeneous symplectic manifold with respect to the fundamental form F and $\mathrm{Aut}\,(M, g, J)$.

Canonical forms for the metrics of non-decomposable locally symmetric para-Kähler spaces were obtained by Patterson [**91**].

4.2. Examples of para-Kähler manifolds. (1) [**70**]. The para-Kähler structure on $\mathbf{R}^{2n}$ given by the pseudo-Euclidean inner product

$\langle\, ,\, \rangle$ and the almost product structure J_{can} defined by

$$\langle\, ,\, \rangle = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}, \qquad J_{\text{can}} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

where both matrices are taken with respect to the canonical basis of $\mathbf{R}^{2n}$, is called the *canonical para-Kähler structure on* $\mathbf{R}^{2n}$.

(2) A *bilagrangian symplectic manifold* is a C^∞ manifold endowed with a closed 2-form F, a pseudo-Riemannian metric g and a couple of supplementary integrable distributions $\mathcal{D}_+$ and $\mathcal{D}_-$, which are isotropic with respect to the metric g, that is, $F|_{\mathcal{D}_+} = 0$ and $F|_{\mathcal{D}_-} = 0$.

Such a manifold is said to be a *parallel bilagrangian symplectic manifold* if $\nabla F = 0$, where ∇ denotes the Levi-Civita connection of g. The relation of the above definitions with the definitions of almost para-Kähler and Kähler manifolds is clear.

(3) [**47**]. Para-Kähler manifolds appear in a natural way in the geometry of negatively curved manifolds. Suppose that N is a simply connected complete Riemannian manifold with sectional curvature $K \leq -1$. Then the unit tangent bundle $S(N)$ of N is fibered over the space M of geodesics of N so that each fiber is an orbit of the geodesic flow of N. The exterior derivative $d\theta$ of the canonical contact form of $S(N)$, which is invariant by the geodesic flow, is pushed forward to a symplectic form F of M by the fiber bundle $S(N) \to M$.

On the other hand, we have that, for a closed Riemannian manifold N of negative curvature, the geodesic flow defined in the unit tangent bundle $S(N)$ is an Anosov flow. Then the splitting $S(N) = E^- \oplus E^0 \oplus E^+$, called the *Anosov splitting* associated with the geodesic flow ϕ_t of N, determines two foliations $\mathcal{E}^-$ and $\mathcal{E}^+$ called the (strongly) stable and unstable foliations of ϕ_t.

In the case of complete simply connected Riemannian manifolds with sectional curvature ≤ -1, the stable and unstable foliations $\mathcal{E}^+$ and $\mathcal{E}^-$ of $S(N)$, descend to foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ of M, which are transverse Lagrangian foliations of (M, F). We thus have an almost para-Kähler manifold, namely $(M, F, \mathcal{F}^+, \mathcal{F}^-)$, associated with the negatively curved manifold N. Moreover, when N is the universal covering of a closed Riemannian manifold whose Anosov splitting is C^∞, the Lagrangian foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ are smooth, and M is para-Kähler.

(4) [**23**]. The almost para-Hermitian structure (G, J) on the tangent bundle TM of a Riemannian manifold (M, g) endowed with a linear connection ∇ is almost para-Kähler if and only if the pair (g, ∇) is cotorsionless; that is, its cotorsion τ vanishes everywhere. We recall that the *cotorsion* is defined in [**22**] as

$$\tau(X, Y, Z) = (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) + g(T(X, Y), Z), \quad X, Y, Z \in \mathfrak{X}(M),$$

T being the torsion of ∇. The structure (G, J) is para-Kähler if and only if (g, ∇) has both vanishing cotorsion and curvature. Notice that J coincides with P in (2.1) and G is the horizontal lift of g with respect to ∇. As for the cotorsion, it coincides with Dg, the exterior covariant differential of g (see Section 2.3).

5. Para-Hermitian symmetric spaces and para-Hermitian homogeneous spaces. Para-Hermitian symmetric spaces are a subfamily of pseudo-Riemannian symmetric spaces. As Hilgert, 'Olafsson and Ørsted have proved, there is a narrow link between symmetric spaces of Hermitian type—introduced by 'Olaffson and Ørsted [**83**] and independently by Matsumoto [**73**], and classified by Doi [**28**]—and with Ol'shankii's regular symmetric spaces [**82**]. As a general result, we have that all the para-Hermitian symmetric spaces are diffeomorphic to the cotangent bundle of another Riemannian symmetric space, which is, in some cases, a Hermitian symmetric space (see Table 2).

Kaneyuki and Kozai have introduced para-Hermitian symmetric spaces in [**55**]. They give many definitions and obtained a lot of results on them. For the sake of brevity we shall only recall a few of them.

5.1. Definitions and first results.

Definition 5.1 [**55**]. A connected almost para-Hermitian manifold (M, g, J) is said to be a *para-Hermitian symmetric space*, if for each point $p \in M$ there exists a paraholomorphic isometry $s_p \in \operatorname{Aut}(M, g, J)$, called the *symmetry* at p, such that

(1) $s_p^2 = \mathrm{id}$.

(2) p is an isolated fixed point of s_p.

Proposition 5.2. *Any para-Hermitian symmetric space* (M, g, J) *is homogeneous para-Kähler, and hence homogeneous symplectic.*

Definition 5.3 [**55**]. Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be a symmetric triple and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the eigenspace decomposition induced by σ. Suppose that $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies the following condition (C):

(C) There exists a linear endomorphism J_0 on $\mathfrak{m}$ and a non-degenerate symmetric bilinear form $\langle\, ,\, \rangle$ on $\mathfrak{m}$ such that

(1) $J_0^2 = \mathrm{id}$.

(2) $[J_0, \mathrm{ad}_{\mathfrak{m}}\mathfrak{h}] = 0$.

(3) $\langle J_0X, Y\rangle + \langle X, J_0Y\rangle = 0, \quad X, Y \in \mathfrak{m}$.

(4) $\langle(\mathrm{ad}\, X)Y_1, Y_2\rangle + \langle Y_1, (\mathrm{ad}\, X)Y_2\rangle = 0, \quad X \in \mathfrak{h}, \quad Y_1, Y_2 \in \mathfrak{m}$.

Then $\{\mathfrak{g}, \mathfrak{h}, \sigma, J_0, \langle\, ,\, \rangle\}$ is called a *para-Hermitian symmetric system.* If, moreover, the pair $\{\mathfrak{g}, \mathfrak{h}\}$ is effective, then it is called an *effective para-Hermitian symmetric system.* We recall that the pair $(\mathfrak{g}, \sigma)$ is called *effective* if the representation $\mathrm{ad}_{\mathfrak{q}}\colon \mathfrak{h} \to \mathrm{End}\,(\mathfrak{q})$ given by $X \mapsto \mathrm{ad}\,(X)|_{\mathfrak{q}}$ is faithful.

Proposition 5.4 [**55**]. *Let* $\{\mathfrak{g}, \mathfrak{h}, \sigma, J_0, \langle\, ,\, \rangle\}$ *be an effective semisimple, that is,* $\mathfrak{g}$ *is semisimple, para-Hermitian symmetric system. Then there exists a unique element* $Z_0 \in \mathfrak{h}$ *such that* $\mathfrak{h}$ *is the centralizer* $\mathfrak{c}(Z_0)$ *of* Z_0 *in* $\mathfrak{g}$ *and* $J_0 = \mathrm{ad}_{\mathfrak{m}} Z_0$.

Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple. Consider the following condition (C′):

(C′) There exists an element $Z \in \mathfrak{g}$ such that $\mathrm{ad}\, Z$ is a semisimple operator having only real eigenvalues and such that $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(Z)$.

Kaneyuki and Kozai proved that conditions (C) and (C′) are equivalent, and the following:

Theorem 5.5. *Let* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ *be an effective semisimple symmetric triple. Then*

(1) *Let* G/H *be a coset space associated with the triple. Suppose that* G/H *is a para-Hermitian symmetric coset space. Then* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$

satisfies condition (C$'$) *and* H *is an open subgroup of the centralizer* $C(Z)$ *in* G. *Conversely,*

(2) *Suppose* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ *satisfies* (C$'$). *Then there exists a connected Lie group* G *with Lie algebra* $\mathfrak{g}$ *such that the coset space* $G/C(Z)$ *is associated with* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$, *where* $C(Z)$ *is the centralizer of* Z *in* G. *Furthermore, for an arbitrary open subgroup* H *of* $C(Z)$, *the coset space* G/H *is a para-Hermitian symmetric space. Moreover, there exists a covering manifold* M_0 *of a symmetric* R*-space such that* $M = G/H$ *is diffeomorphic to the cotangent bundle* T^*M_0 *of* M_0.

5.2. Classification and structure of semisimple para-Hermitian symmetric spaces. Every para-Hermitian symmetric space with semisimple group is diffeomorphic to the cotangent bundle of a covering manifold of a Riemannian symmetric space of a particular type, called R-symmetric spaces. So the para-Hermitian symmetric spaces are candidates to be phase spaces of dynamical systems. Since we have, for instance, the phase space $T^*(\mathrm{SO}\,(3))$ of the rigid solid, we are probably faced with important physical situations. The infinitesimal classification of para-Hermitian symmetric spaces with semisimple group up to paraholomorphic equivalence is obtained in [**55**] and [**48**], and in [**55**] Table 2 is given.

In the following list, $G_{m,n}(\mathbf{F})$ denotes the Grassmann manifold of m-planes in $\mathbf{F}^{m+n}$, where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. $Q_{m,n}(\mathbf{R})$ denotes the real quadric in $P_{m+n-1}(\mathbf{R})$ defined by the quadratic form of signature (m, n). $Q_n(\mathbf{C})$ denotes the complex quadric in $P_{n+1}(\mathbf{C})$. $P_2(\mathbf{O})$ denotes the octonion projective plane. The list on the right of the table contains those R-symmetric spaces M_0^* with the property that if $M = G/H$ is a para-Hermitian symmetric space corresponding to the symmetric pair $(\mathfrak{g}, \mathfrak{h})$ associated with the specific M_0^*, then M is diffeomorphic to the cotangent bundle T^*M_0 of a covering manifold M_0 of M_0^*. Note the six Hermitian symmetric spaces, appearing on the right.

Remark 5.6. We recall here the following definitions. Let G, respectively $\tilde{G}$, be a reductive irreducible real, respectively connected complex, algebraic group. The quotient space $M = G/U$, respectively

TABLE 2. Para-Hermitian symmetric simple Lie algebras.

$(\mathfrak{g},\mathfrak{h})$	M_0^*
$(\mathfrak{sl}(m+n,\mathbf{R}),\mathfrak{sl}(m,\mathbf{R})+\mathfrak{sl}(n,\mathbf{R})+\mathbf{R})$	$G_{m,n}(\mathbf{R})$
$(\mathfrak{sl}(m+n,\mathbf{C}),\mathfrak{sl}(m,\mathbf{C})+\mathfrak{sl}(n,\mathbf{C})+\mathbf{C})$	$G_{m,n}(\mathbf{C})$
$(\mathfrak{su}^*(2m+2n),\mathfrak{su}^*(2m)+\mathfrak{su}^*(2n)+\mathbf{R})$	$G_{m,n}(\mathbf{H})$
$(\mathfrak{su}(n,n),\mathfrak{sl}(n,\mathbf{C})+\mathbf{R})$	$\mathrm{U}(n)$
$(\mathfrak{so}(n,n),\mathfrak{sl}(n,\mathbf{R})+\mathbf{R})$	$\mathrm{SO}\,(n)$
$(\mathfrak{so}^*(4n),\mathfrak{su}^*(2n)+\mathbf{R})$	$\mathrm{U}(2n)/\mathrm{Sp}\,(n)$
$(\mathfrak{so}(2n,\mathbf{C}),\mathfrak{sl}(n,\mathbf{C})+\mathbf{C})$	$\mathrm{SO}\,(2n)/\mathrm{U}(n)$
$(\mathfrak{so}(m+1,n+1),\mathfrak{so}(m,n)+\mathbf{R})$	$Q_{m+1,n+1}(\mathbf{R})$
$(\mathfrak{so}(n+2,\mathbf{C}),\mathfrak{so}(n,\mathbf{C})+\mathbf{C})$	$Q_n(\mathbf{C})$
$(\mathfrak{sp}(n,\mathbf{R}),\mathfrak{sl}(n,\mathbf{R})+\mathbf{R})$	$\mathrm{U}(n)/\mathrm{O}(n)$
$(\mathfrak{sp}(n,n),\mathfrak{su}^*(2n)+\mathbf{R})$	$\mathrm{Sp}\ \ (n)$
$(\mathfrak{sp}(n,\mathbf{C}),\mathfrak{sl}(n,\mathbf{C})+\mathbf{C})$	$\mathrm{Sp}\,(n)/\mathrm{U}(n)$
$(E_6^1,\mathfrak{so}(5,5)+\mathbf{R})$	$G_{2,2}(\mathbf{H})/\mathbf{Z_2}$
$(E_6^4,\mathfrak{so}(1,9)+\mathbf{R})$	$P_2(\mathbf{O})$
$(E_6^{\mathbf{C}},\mathfrak{so}(10,\mathbf{C})+\mathbf{C})$	$E_6/\mathrm{Spin}\,(10)\cdot T^1$
$(E_7^1,E_6^1+\mathbf{R})$	$\mathrm{SU}\,(8)/\mathrm{Sp}\,(4)\cdot\mathbf{Z_2}$
$(E_7^3,E_6^4+\mathbf{R})$	$T^1\cdot E_6/F_4$
$(E_7^{\mathbf{C}},E_6^{\mathbf{C}}+\mathbf{C})$	$E_7/E_6\cdot T^1$

$\tilde{M}=\tilde{G}/\tilde{U}$, by a parabolic subgroup U, respectively $\tilde{U}$, of G, respectively $\tilde{G}$, is called an R-space, respectively complex R-space. For the definition of symmetric R-space, which involves Dynkin diagrams, see, for instance, [**115**, p. 82].

The structure of the (simple) group G of a para-Hermitian symmetric space $M=G/H$ is studied in [**50**], and the isotropy group of a para-Hermitian symmetric space in [**56**].

As is well-known, M. Flensted-Jensen's knowledge of both the theories of group representations and semisimple symmetric spaces, has permitted him to emphasize the important role of the affine symmetric spaces in the theory of group representations (see [**13, 31, 87**]). In this spirit, Kaneyuki studies in [**49**] the orbit structure of compactifications of para-Hermitian symmetric spaces. We recall here the example given by him, which gives an idea of the general situation: Let $\mathcal{H}$ be the hyperboloid of revolution in $\mathbf{R}^3$ given by the equation $x^2+y^2-z^2=1$. $\mathcal{H}$ is viewed as the cotangent bundle of the real projective space $P_1(\mathbf{R})$. $\mathcal{H}$ can be seen as the affine symmetric space $\mathrm{SL}(2,\mathbf{R})/\mathbf{R}^*$, where $\mathbf{R}^*$ is identified with the subgroup of diagonal matrices in $\mathrm{SL}(2,\mathbf{R})$. The $\mathrm{SL}(2,\mathbf{R})$-action on $\mathcal{H}$ leaves invariant each of the two families L_1 and L_2 of generatrices of $\mathcal{H}$. Through an arbitrary point $p \in \mathcal{H}$ there pass two generating lines $l_i \in L_i$ $(i=1,2)$, which meet the line of stricture of $\mathcal{H}$, viewed as $P_1(\mathbf{R})$, in two points q_i. By assigning the pair (q_1,q_2) to the point $p \in \mathcal{H}$ we have an embedding of $\mathcal{H}$ into the 2-torus $T^2 = P_1(\mathbf{R}) \times P_1(\mathbf{R})$. The embedding maps L_1 or L_2 into the meridians or the parallels on T^2, respectively. The $\mathrm{SL}(2,\mathbf{R})$-action on $\mathcal{H}$ is hence transferred to the action on T^2 of the diagonal subgroup of $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$. An elementary argument shows that the torus T^2 is decomposed into two $\mathrm{SL}(2,\mathbf{R})$-orbits: one is $\mathcal{H}$ and the other is a 1-dimensional orbit diffeomorphic to $P_1(\mathbf{R})$. In [**50**], this phenomenon is generalized to higher dimensions: *Let $M = G/H$ be a para-Hermitian symmetric space, which is diffeomorphic to the cotangent bundle of a covering manifold of a symmetric R-space M_0^*. Let $\tilde{M} = M_0^* \times M_0^*$. Then, M is imbedded in $\tilde{M}$ as a single orbit through the origin of $\tilde{M}$ under the action of the diagonal subgroup of $G \times G$. Thus, $\tilde{M}$ can be viewed as a compactification of M.* In [**50**] the orbit structure of $\tilde{M}$ is studied. It turns out that it is somewhat similar to the structure of the closure of an irreducible bounded symmetric domain under its holomorphic automorphism group [**65**]. This situation can be extended to the case of para-Hermitian homogeneous spaces.

5.2. Para-Hodge manifolds.

Definition 5.7 [**58**]. A para-Kähler manifold (M,g,J) is called a *para-Hodge manifold* if the cohomology class $[F]$ of its fundamental 2-form F is an integral class in $H^2(M,\mathbf{R})$.

Example 5.8 [**58**]. A para-Hermitian symmetric space with second Betti number $b_2=0$ is always para-Hodge. Let M be the cotangent bundle T^*M_0 over a symmetric R-space M_0. Then M is a para-

Hermitian symmetric coset space of a semisimple Lie group. The para-Kähler metric g of M is then induced by the Killing form of the Lie algebra of G. If M_0 is one of the group manifolds $\mathrm{SO}\,(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}\,(n)$, $\mathrm{U}(2n)/\mathrm{Sp}\,(n)$, or the sphere S^n, $n \geq 2$, then the second Betti number of M vanishes, and so g is para-Hodge.

Example 5.9 [**58**]. Let (M, g, J) be a para-Hermitian symmetric space with simple automorphism group $\mathrm{Aut}\,(M, g, J)$, where g is induced from the Killing form of G. Then (M, g, J) is para-Hodge with $b_1(M) = 0$ if and only if M is the cotangent bundle of a covering manifold of a symmetric R-space M_0^*, which is not the Silov boundary of an irreducible symmetric bounded domain.

In [**58**], some criteria for a para-Hermitian symmetric space to be para-Hodge and many examples are given.

5.4. Semisimple para-Hermitian symmetric spaces as quantizable coadjoint orbits.

Definition 5.10. A symplectic manifold (M, ω) is said to be *quantizable* if the cohomology class $[\omega]$ in $H^2(M, \mathbf{R})$ of the nondegenerate closed 2-form ω lies in the image $i^* H^2(M, \mathbf{Z})$ of the homomorphism $i^*: H^2(M, \mathbf{Z}) \to H^2(M, \mathbf{R})$ induced by the inclusion map $i: \mathbf{Z} \hookrightarrow \mathbf{R}$.

The following theorem gives a useful criterion in order to know whether a symplectic manifold is quantizable.

Theorem 5.11 [**66**]. *A symplectic manifold (M, ω) is quantizable if and only there is a Hermitian line bundle $L \to M$ over M with an invariant connection ∇ such that its curvature form R_∇ satisfies $(1/2\pi i) R_\nabla = \omega$.*

Definition 5.12. Let G be any Lie group. Then G acts on the (real) dual space $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$ via the contragredient of the adjoint representation by

$$(s \circ f)(X) = f(\mathrm{Ad}\,(s^{-1})X), \quad s \in G, \quad X \in \mathfrak{g}, \quad f \in \mathfrak{g}^*. \tag{5.1}$$

Each orbit in $\mathfrak{g}^*$ induced by this action is called a *coadjoint orbit*.

Proposition 5.13 [**61**]. *Each coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$ of the action of the Lie group G on $\mathfrak{g}^*$ is endowed with a natural closed 2-form ω_0 such that $(\mathcal{O}, \omega_0)$ is a symplectic manifold.*

Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple and let $Z \in \mathfrak{g}$ such that

(5.2) (1) $\mathfrak{h} = \{X \in \mathfrak{g} : [X, Z] = 0\}$,

(5.3) (2) $\operatorname{ad} Z \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ has only real eigenvalues.

By Theorem 5.5, $G/C(Z)$ is a para-Hermitian symmetric coset space. If B denotes the Killing form of $\mathfrak{g}$, the associated natural isomorphism $\flat \colon \mathfrak{g} \to \mathfrak{g}^*$ is given by $X \mapsto f_X$, where $f_X(Y) = B(Y, X)$ for $Y \in \mathfrak{g}$. This isomorphism intertwines the adjoint action of G on G and the coadjoint action of G on $\mathfrak{g}^*$ given in (5.1). Thus $\mathcal{O}_Z = G/C(Z)$ is a coadjoint orbit. Then we have

Theorem 5.14 [**57**]. *Let $\mathcal{O}_Z = G/C(Z)$ be the para-Hermitian symmetric coadjoint orbit associated with the effective semisimple triple $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ which satisfies* (5.2) *and* (5.3), *and let ω_Z be the Kirillov structure on $\mathcal{O}_Z$* (*Proposition* 5.13). *Suppose G is simple and does not admit a complex Lie algebra structure, and suppose $C(Z)$ is connected. Then $(\mathcal{O}_Z = G/C(Z), \omega_Z)$ is quantizable. Moreover, $G/C(Z)$ is a para-Hodge manifold.*

Theorem 5.15 [**57**]. *Let $(\mathfrak{g}, \mathfrak{h})$ be one of the following symmetric pairs:* $(\mathfrak{su}^*(2m+2n), \mathfrak{su}^*(2m) + \mathfrak{su}^*(2n) + \mathbf{R})$, (the quaternionic case); $(\mathfrak{so}(n+1,1), \mathfrak{so}(n) + \mathbf{R})$ *for* $n \geq 2$, (the n-sphere case); $(\mathfrak{sp}(n,n), \mathfrak{su}^*(2n) + \mathbf{R})$, (the Sp (n)-case); *and* $(E_6^4, \mathfrak{so}(1,9) + \mathbf{R})$, (the octonion case). *Then $(\mathfrak{g}, \mathfrak{h})$ is part of an effective simple triple* (*i.e.,* $\mathfrak{g}$ *is simple*), *which satisfies* (5.2) *and* (5.3) *for a suitable $Z \in \mathfrak{g}$ such that the orbit $\mathcal{O}_Z$ in Theorem* 5.14 *is simply connected. In particular $C(Z)$ is connected and all the conclusions of Theorem* 5.14 *apply to* $(\mathcal{O}_Z, \omega_Z)$.

5.5. Para-Hermitian homogeneous spaces. Para-Hermitian homogeneous spaces have been introduced and studied in [**51**] (see also [**52,54**]). Kaneyuki obtains a classification in terms of a natural number ν in such a way that para-Hermitian symmetric spaces correspond to the case $\nu = 1$. This number is associated with the kind of the GLA (graded Lie algebra) involved. He studies the relations between three Lie-algebraic objects: para-Kähler algebras, dipolarizations and graded Lie algebras, and gives the infinitesimal classification of semisimple para-Kähler coset spaces of the second kind in [**51, 53**].

6. Para-Kähler space forms.

6.1. Para-Kähler manifolds of constant paraholomorphic sectional curvature. Consider the tensor field R' on the para-Kähler manifold (M, g, J) defined by

$$\begin{aligned} R'(X,Y,Z,W) = \frac{1}{4}\{ & g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\ & - g(X,JZ)g(Y,JW) + g(X,JW)g(Y,JZ) \\ & - 2g(X,JY)g(Z,JW)\}, \quad X,Y,Z,W \in \mathfrak{X}(M). \end{aligned}$$

This tensor field was independently defined in [**93**] and [**34**]. Let $H(X)$ denote the paraholomorphic sectional curvature defined by a vector X. We have the following

Theorem 6.1 [**34**]. *Let (M, g, J) be a para-Kähler manifold such that for each $x \in M$, there exists $c_x \in \mathbf{R}$ satisfying $H(X) = c_x$ for every $X \in T_xM$ such that $g(X,X)g(JX,JX) \neq 0$. Then the Riemann-Christoffel tensor R satisfies $R = cR'$, where c is the function defined by $x \mapsto c_x$, and conversely.*

Definition 6.2. A para-Kähler manifold (M, g, J) is said to be of *constant paraholomorphic sectional curvature* c if it satisfies the conditions of the previous theorem.

One has the following Schur-type result:

Theorem 6.3 [**34**]. *Let (M, g, J) be a para-Kähler manifold with* *constant paraholomorphic sectional curvature* c*. If* $\dim M > 2$*, then* c *is a constant function.*

Consequently, if $\dim M = 2$, in order to guarantee that a para-Kähler manifold has constant paraholomorphic sectional curvature c, one must assume that c is a constant function.

6.2. The paracomplex projective models $P_n(B)$. The paracomplex projective spaces, $P_n(\mathbf{A})$, were introduced by Libermann [**70**] and Rozenfeld [**108, 109**, p. 578]. We recall that in the complex case one can—see, for instance, [**16**]—normalize the complex homogeneous coordinate vectors Z, putting $Z_0 = Z/(Z,Z)^{1/2}$ and thus give to the Fubini-Study metric on the complex projective space $P_n(\mathbf{C})$ the expression $ds^2 = (dZ_0, dZ_0) - (dZ_0, Z_0)(Z_0, dZ_0)$. In [**70**, p. 89], the author uses a similar procedure and obtains a pseudo-Riemannian metric of "Fubini-Study" type $ds^2 = (de_0, de'_0) - (e_0, de'_0)(e'_0, de_0)$. This expression is valid only locally, in the open subset of the space $P_n(\mathbf{A}) \approx P_n(\mathbf{R}) \times P_n(\mathbf{R})$ complementary of a singular hyperquadric.

The paracomplex projective models $P_n(B)$ were introduced in [**34**]. They are not diffeomorphic to Rozenfeld-Libermann's paracomplex projective space, but they are, for $n > 1$, models of para-Kähler manifolds of nonvanishing constant paraholomorphic curvature, as is proved in [**34**]. Notice that, since the metric has signature (n, n), in order to change the sign of the sectional curvature it suffices changing the sign of the metric. As they are, moreover, a kind of projective space, we shall call them the *paracomplex projective models*. They are also pseudo-Riemannian symmetric spaces [**12**]; specifically, they are para-Hermitian symmetric spaces [**55**] and, moreover, harmonic symmetric spaces [**3**].

In [**35**], it is shown that *the spaces* $P_n(B)$ *must be considered as one of the most natural homogeneous pseudo-Riemannian spaces*. In fact, it is shown that in order to obtain the geometry of these spaces, it suffices only giving a real finite dimensional vector space and its dual space, as we now explain.

Let E be a real $(n+1)$-dimensional vector space, and E^* its dual space. On the space $E \oplus E^*$ there exist:

(1) A natural nondegenerate bilinear form $\langle\, ,\, \rangle$ given by

$$\langle x+\alpha, y+\beta\rangle = (2/c)(\alpha(y)+\beta(x)),$$
$$x, y \in E, \quad \alpha, \beta \in E^*, \quad 0 \neq c \in \mathbf{R}.$$

(2) A $(1,1)$ tensor J_0 such that $J_0|_E = \mathrm{id}_E$, $J_0|_{E^*} = -\mathrm{id}_{E^*}$.

The subgroup of $\mathrm{GL}(E \oplus E^*)$ which preserves $\langle\, ,\, \rangle$ and J_0 can be identified with $\mathrm{GL}(E)$. We introduce in

$$(E \oplus E^*)_+ = \{x+\alpha \in E \oplus E^* : \langle x+\alpha, x+\alpha\rangle = (4/c)\alpha(x) > 0\}$$

the following equivalence relation $\sim$: $x+\alpha \sim ax+b\alpha$, $a>0$, $b>0$, and define

$$P(E \oplus E^*) = (E \oplus E^*)_+/\sim .$$

The identity component $\mathrm{GL}_0(E)$ of $\mathrm{GL}(E)$ acts transitively on the pseudosphere $S = \{x+\alpha \in E \oplus E^* : \alpha(x) = 1\}$ and also on $P(E \oplus E^*)$, making it a homogeneous manifold $(P(E \oplus E^*), \mathrm{GL}_0(E))$ and the base space of a fiber bundle $p\colon S \to P(E \oplus E^*)$ with fiber $\mathbf{R}$ and such that the subgroup $\{aI \in \mathrm{GL}_0(E) : a > 0\}$ of $\mathrm{GL}_0(E)$ acts transitively on the fibers. From this bundle we can endow $P(E \oplus E^*)$ with a pseudo-Riemannian metric $\langle\, ,\, \rangle$ and an almost product structure J induced very simply via S from the structures in $E \oplus E^*$, which, as it is proved, make $(P(E \oplus E^*), \mathrm{GL}_0(E))$ a para-Kähler space form *isomorphic to the paracomplex projective model* $P_n(B)$. Moreover, the construction of $(P(E \oplus E^*), \langle\, ,\, \rangle, J)$ is natural with respect to the category of finite dimensional real vector spaces. In this sense, because of the economy of the initial data, the geometry of these spaces have a right to stand immediately after affine and projective geometry and before, say, the geometry of the sphere. This space is, thus, one of the more natural pseudo-Riemannian homogeneous spaces.

The classification of para-Kähler space forms is given in [**34, 38**]. For the sake of brevity, we do not give the results, but only underline that one has a rich family of para-Kähler space forms, and we are so in a intermediate situation between the very rich family of (pseudo)-Riemannian space forms, and the case of the complex projective space $P_n(\mathbf{C})$, none of whose space forms is even Hermitian [**119**].

7. Some applications.

7.1. Nonsymmetric gravitational theory. A new theory of gravitation is proposed in [**78**], called *nonsymmetric gravitational theory* because the author considers that the geometry of space-time is determined by a nonsymmetric tensor $g_{\mu\nu}$. The author wants thus to overcome the difficulties of general relativity at the singularities of collapsed stars and cosmology. The geometrical interpretation of this theory is given in [**68**] (see Crumeyrolle [**26**]), where it is shown that one can obtain the theory by considering a four-dimensional space-time M, but enlarging the tangent space at each point by a procedure analogous to the complexification of the tangent bundle, which we may call paracomplexification, for it is done using paracomplex numbers.

7.2. Chronogeometry and causal symmetric spaces. In [**82**], 'Olafsson, motivated by chronogeometry and other applications, considers the general problem of the classification of all the *infinitesimal causal manifolds*. He makes the simple but strong restriction of considering only symmetric spaces, and poses the following triple problem:

(1) Classify all symmetric spaces (G, H, σ) such that the (-1)-eigenspace of the differential of the involution σ, admits H-invariant cones, being H the isotropy.

(2) Given such a triple (G, H, σ), classify all the possible invariant cones.

(3) (Global problem.) Which of these invariant cones do actually arise from a global ordering on the space G/H?

He proves that the classification of irreducible semisimple causal spaces is reduced to three cases, and he also classifies the H-invariant regular cones.

7.3. Mechanics. Isotropic subbundles and Lagrangian subbundles are important in theoretical Mechanics. One can find contributions to Mechanics in [**11, 86**].

7.4. Gauge fields. Zhong [**123**] considers the hyperbolic complex linear groups and their relation with the real linear groups, and applies these results to the relation between the real and the complex local gauge symmetries.

7.5. Unitary field theories. Crumeyrolle [**26**] has used para-Hermitian manifolds to geometrize and generalize the Einstein-Schrödinger unitary field theories.

7.6. Strings. Jensen and Rigoli prove in [**46**] that, under certain conditions, an almost para-Hermitian maps is a string, and characterize the strings which are almost para-Hermitian.

7.7. Representation theory [**81**]. The conditions for a semisimple symmetric space M to be of Hermitian type are exactly the right ones to provide the existence of Hardy spaces and holomorphic discrete series associated with M (see [**44**]).

7.8. Causal spaces [**82**, p. 25]. The semisimple symmetric orbits in a Lie algebra $\mathfrak{g}$ are not only interesting from the geometric point of view but also in representation theory, according to Kirillov's orbit philosophy. Furthermore, many interesting representations are living on function spaces and cohomology spaces on the Hermitian spaces. They are also related to graded Lie algebras and Jordan algebras. It also turns out that causal spaces are built up from such orbits by the dual/associated construction and that the classification and geometry of these spaces can be obtained from results on complex or paracomplex spaces.

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REFERENCES

1. F.S. Ahrarov, *Analytical surfaces with constant curvature in the spaces of* 0-*couples*, Izv. Mat. **5** (1978), 127–130 (Russian).

2. F.S. Ahrarov and A.P. Norden, *Intrinsic geometry of analytical surfaces in the space of non-degenerate* 0-*couples*, Izv. Mat. **8** (1978), 19–30 (Russian).

3. C. Allamigeon, *Espaces homogènes symétriques à groupe semi-simple*, C.R. Acad. Sci. Paris Sér. I Math. **243** (1956), 121–123.

4. F. Amato, *CR-sous-variétés co-isotropes inducées dans une C^∞-variété pseudo-riemannienne neutre*, Boll. Un. Mat. Ital. A **4** (1985), 433–440.

5. W. Ambrose and I.M. Singer, *On homogeneous manifolds*, Duke Math. J. **25** (1958), 647–669.

6. G. Arca, *Variétés pseudo-riemanniennes structurées par une connexion spin-euclidienne et possédant la propiété de Killing*, C.R. Acad. Sci. Paris Sér. I Math. **290** (1980), 839–842.

7. G. Arca, R. Caddeo and R. Rosca, *Variétés para-kähleriennes possédant la propriété concirculaire*, C.R. Acad. Sci. Paris Sér. I Math. **286** (1978), 1209–1212.

8. C. Bejan, *A classification of the almost parahermitian manifolds*, Proc. Conference on Diff. Geom. and Appl., Dubrovnik, (1988), 23–27.

9. ———, *Almost parahermitian structures on the tangent bundle of an almost paracohermitian manifold*, Proc. Fifth Nat. Sem. Finsler and Lagrange spaces, Brasov, (1988), 105–109.

10. ———, *Structuri hiperbolice pe diverse spatii fibrate*, Ph.D. thesis, Iaşi, 1990.

11. A. Bejancu and T. Otsuki, *General Finsler connections on a Finsler vector bundle*, Kodai Math. J. **10** (1987), 143–152.

12. M. Berger, *Les espaces symétriques non compacts*, Bull. Soc. Math. **74** (1957), 85–177.

13. F. Bien, *D-modules and spherical representations*, Math. Notes **39**, Princeton Univ. Press, 1990.

14. I. Bouzon, *Structures presque cohermitiennes*, C.R. Acad. Sci. Paris Sér. I Math. **258** (1964), 412–415.

15. É. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. **54** (1926), 214–264.

16. S.S. Chern, *Einstein hypersurfaces in a Kählerian manifold of constant holomorphic curvature*, J. Differential Geometry **1** (1967), 21–31.

17. W.K. Clifford, *A preliminary sketch on biquaternions*, Proc. London Math. Soc. **4** (1873), 381–395. Reprinted in: *Mathematical papers*, Chelsea Publ., New York, 1968.

18. ———, *On the theory of screws in a space of constant curvature*, id., 402–405.

19. ———, *On the motion of a body in a elliptic space*, 1874, id., 378–384.

20. ———, *Further note on biquaternions*, id., 385–396.

21. V. Cruceanu, *Connexions compatibles avec certaines structures sur un fibré vectoriel banachique*, Czechoslovak Math. J. **24** (1974), 126–142.

22. ———, *Certaines structures sur le fibré tangent*, Proc. Inst. Math. Iaşi, Ed. Acad. Rom. (1976), 41–49.

23. ———, *Une structure parakählerienne sur le fibré tangent*, Tensor (N.S.) **39** (1982), 81–84.

24. ———, *Sur certains morphismes des structures géométriques*, Rend. Mat. Appl. **6** (1986), 321–332.

25. ———, *Une classe de structures géométriques sur le fibré cotangent*, Tensor (N. S.), to appear.

26. A. Crumeyrolle, *Variétés différentiables à coordonnées hypercomplexes. Application à une géométrisation et à une généralisation de la théorie d'Einstein-Schrödinger*, Ann. Fac. Sci. Toulouse **26** (1962), 105–137.

27. J.E. D'Atri and H.K. Nickerson, *The existence of special orthonormal frames*, J. Differential Geom. **2** (1968), 393–409.

28. H. Doi, *A classification of certain symmetric Lie algebras*, Hiroshima Math. J. **11** (1981), 173–180.

29. C. Ehresmann, *Sur les variétés presque complexes*, Act. Proc. Intern. Cong. Math., (1950), 412–419.

30. M. Flensted-Jensen, *Spherical functions on a real semisimple group. A method of reduction to the complex case*, J. Funct. Anal. **30** (1978), 106–146.

31. ———, *Discrete series for semisimple symmetric spaces*, Ann. of Math. **111** (1980), 253–311.

32. H. Freudenthal, *Lie groups in the foundations of geometry*, Adv. Math. **1** (1970), 145–190.

33. A. Fujimoto, *Theory of G-structures*, Study Group of Geom., 1972.

34. P.M. Gadea and A. Montesinos Amilibia, *Spaces of constant paraholomorphic sectional curvature*, Pacific J. Math. **136** (1989), 85–101.

35. ———, *The paracomplex projective spaces as symmetric and natural spaces*, Indian J. Pure Appl. Math. **23** (1992), 261–275.

36.———, *Totally umbilical pseudo-Riemannian submanifolds of the paracomplex projective space*, Czechoslovak Math. J. **44** (1994), 741–756.

37. P.M. Gadea and J. Muñoz Masqué, *Classification of almost para-Hermitian manifolds*, Rend. Mat. Appl. **11** (1991), 377–396.

38. ———, *Classification of homogeneous parakählerian space forms*, Nova J. Algebra Geom. **1** (1992), 111–124.

39. P.M. Gadea and J.A. Oubiña, *Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures*, Houston J. Math. **18** (1992), 449–465.

40. V.V. Goldberg and R. Rosca, *Biconformal vector fields on manifolds endowed with a certain differential conformal structure*, Houston J. Math. **14** (1988), 81–95.

41. J.T. Graves, *On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables*, Phil. Magaz., London-Edinburgh-Dublin, **26** (1845), 315–320.

42. A. Gray and L.M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

43. J. Hilgert, *The hyperboloid as ordered symmetric space*, Sem. Sophus Lie **1** (1991), 135–142.

44. J. Hilgert, G. 'Olafsson and B. Ørsted, *Hardy spaces associated to symmetric spaces of Hermitian type*, Math. Gottingensis, Heft **29**, 1989.

45. S. Ianuş and R. Rosca, *Variétés para-Kähleriennes structurées par une connexion géodesique*, C.R. Acad. Sci. Paris Sér. I Math. **280** (1975), 1621–1623.

46. G. Jensen and M. Rigoli, *Neutral surfaces in neutral four-spaces*, Matematiche (Catania) **45** (1991), 407–443.

47. M. Kanai, *Geodesic flows of negatively curved manifolds with stable and unstable flows*, Ergodic Theory Dynamical Systems **8** (1988), 215–239.

48. S. Kaneyuki, *On classification of parahermitian symmetric spaces*, Tokyo J. Math. **8** (1985), 473–482.

49. ———, *On orbit structure of compactifications of parahermitian symmetric spaces*, Japan. J. Math. **13** (1987), 333–370.

50. S. Kaneyuki, *A decomposition theorem for simple Lie groups associated with parahermitian symmetric spaces*, Tokyo J. Math. **10** (1987), 363–373.

51. ———, *On a remarkable class of homogeneous symplectic manifolds*, Proc. Japan Acad. Math. Sci. **67** (1991), 128–131.

52. ———, *Homogeneous symplectic manifolds and dipolarizations in Lie algebras*, Tokyo J. Math. **15** (1992), 313–325.

53. ———, *On the subalgebras* $\mathfrak{g}_0$ *and* $\mathfrak{g}_{ev}$ *of semisimple graded Lie algebras*, J. Math. Soc. Japan **45** (1993), 1–19.

54. S. Kaneyuki and H. Asano, *Graded Lie algebras and generalized Jordan triple systems*, Nagoya Math. J. **112** (1988), 81–115.

55. S. Kaneyuki and M. Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math. **8** (1985), 81–98.

56. ———, *On the isotropy group of the automorphism group of a parahermitian symmetric space*, Tokyo J. Math. **8** (1985), 483–490.

57. S. Kaneyuki and F. Williams, *On a class of quantizable co-adjoint orbits*, Algebras Groups Geom. **2** (1985), 70–94.

58. ———, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. **99** (1985), 173–187.

59. P.F. Kelly and R.B. Mann, *Ghost properties of algebraically extended theories of gravitation*, Classical Quantum Gravity **3** (1986), 705–712.

60. V.F. Kirichenko, *Tangent bundles from the point of view of generalized Hermitian geometry*, Izv. Vyssh. Uchebn. Zaved. Mat., Math. **7** (1984), 50–58 (Russian); Soviet Math. (Iz. VUZ) **28** (1984), 63–74 (English).

61. A. Kirillov, *Unitary representations of nilpotent groups*, Russian Math. Surveys **17** (1962), 53–104.

62. F. Klein, *Vergleichende Betrachtungen über neuere geometrische Vorschungen*, A. Deichert, Erlangen, 1872.

63. S. Kobayashi and T. Nagano, *Filtered Lie algebras and geometric structures*, I, J. Math. Mech. **13** (1964), 875–908.

64. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Intersc. Publ., 1963 and 1969.

65. A. Korányi and J.A. Wolf, *Realization of Hermitian symmetric spaces as generalized half-planes*, Ann. of Math. **81** (1965), 265–288.

66. B. Kostant, *Quantization and unitary representations. Part* I: *Prequantization*, Lect. notes in Math., **170**, Springer-Verlag, 1970.

67. A.P. Kotelnikov, *Twist calculus and some of its applications to geometry and mechanics*, Kazan, 1895 (Russian).

68. G. Kunstatter and J.W. Moffat, *Geometrical interpretation of a generalized theory of gravitation*, J. Math. Phys. **24** (1986), 886.

69. P. Libermann, *Sur les structures presque paracomplexes*, C.R. Acad. Sci. Paris Sér. I Math. **234** (1952), 2517–2519.

70. ———, *Sur le problème d'équivalence de certaines structures infinitésimales*, Ann. Mat. Pura Appl. (4) **36** (1954), 27–120.

71. R.B. Mann, *New ghost-free extensions of general relativity*, Classical Quantum Gravity **6** (1989), 41–57.

72. G. Markov and M. Prvanović, *π-holomorphically planar curves and π-holomorphically projective transformations*, Publ. Math. Debrecen, **37** (1990), 273–284.

73. S. Matsumoto, *Discrete series for an affine symmetric space*, Hiroshima Math. J. **11** (1981), 53–79.

74. O. Mayer, *Géométrie biaxiale différentielle des courbes*, Bull. Math. Soc. Roum. Sci. **4** (1938), 1–4.

75. ———, *Biaxiale Differentialgeometrie der Kurven und Regelflĕchen*, Anal. Sci. Univ. Jassy **27** (1941), 327–410.

76. O. Mayer, *Geometria biaxiala diferentiala a suprafeţelor*, Lucr. Ses. Şti. Acad. Rep. Pop. Romàne, 1950, 1–7.

77. R. Miron and G. Atanasiu, *Existence et arbitrariété des connexions compatibles à une structure Riemann généralisée du type presque k-horsymplectique métrique*, Kodai Math. J. **6** (1983), 228–237.

78. J.W. Moffat, *New theory of gravitation*, Phys. Rev. **19** (1979), 3554–3558.

79. ———, *Nonsymmetric gravitation theory and its experimental consequences*, World Sci. Publ., Singapore, 1984.

80. ———, *Review of the nonsymmetric gravitational theory*, Summer Inst. on Gravit., Banff Center, Banff, Alberta, Canada, 1990.

81. G. 'Olafsson, *Symmetric spaces of Hermitian type*, Differential Geom. Appl. **1** (1991), 195–233.

82. ———, *Causal symmetric spaces*, Math. Gottingensis, Heft **15**, 1990.

83. G. 'Olafsson and B. Ørsted, *The holomorphic discrete series for affine symmetric spaces*, J. Funct. Anal. **81** (1988), 126–159.

84. G.I. Ol'shanskii, *Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series*, Functional Anal. Appl. **15** (1982), 275–285.

85. Z. Olszak, *On conformally flat parakählerian manifolds*, Math. Balkanica (N.S.) **5** (1991), 302–307.

86. V. Oproiu, *A pseudo-Riemannian structure in Lagrange geometry*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. **33** (1987), 239–254.

87. T. Oshima and T. Matsuki, *A description of discrete series for semisimple symmetric spaces: Group representations and systems of Differential Equations*, Adv. Stud. Pure Math. **4** (1984), 331–390, Kinokuniya, Tokyo and North-Holland, Amsterdam.

88. R.B. Pal and R.S. Mishra, *Hypersurfaces of almost hyperbolic Hermite manifolds*, Indian J. Pure Appl. Math. **11** (1980), 628–632.

89. D. Papuc, *Sur les variétés des espaces kleinéens à groupe linéaire complétement reductible*, C. R. Acad. Sci. Paris Sér. I Math. **256** (1963), 62–64; 589–591.

90. E.M. Patterson, *Riemann extensions which have Kähler metrics*, Proc. Roy. Soc. Edinburgh Sect. A **64** (1954), 113–126.

91. ———, *Symmetric Kähler spaces*, J. London Math. Soc. **30** (1955), 286–291.

92. ———, *A characterization of Kähler manifolds in terms of parallel fields of planes*, J. London Math. Soc. **28** (1958), 260–269.

93. M. Prvanović, *Holomorphically projective transformations in a locally product space*, Math. Balkanica **1** (1971), 195–213.

94. P.K. Rashevskij, *The scalar field in a stratified space*, Trudy Sem. Vektor. Tenzor. Anal. **6** (1948), 225–248.

95. R. Rosca, *Variétés pseudo-riemanniennes $V^{n,n}$ de signature (n,n) et à connexion self-orthogonale involutive*, C. R. Acad. Sci. Paris Sér. I Math. **280** (1974), 959–961.

96. ———, *Quantic manifolds with para-co-Kählerian structures*, Kodai Math. Sem. Rep. **27** (1976), 51–61.

97. ———, *Para-Kählerian manifolds carrying a pair of concurrent self-orthogonal vector fields*, Abh. Math. Sem. Univ. Hamburg **46** (1977), 205–215.

98. ———, *Espace-temps possédant la propriété géodésique*, C.R. Acad. Sci. Paris Sér. I Math. **285** (1977), 305–308.

99. ———, *Sous-variétés anti-invariantes d'une variété parakählerienne structurée par une connexion géodesique*, C.R. Acad. Sci. Paris Sér. I Math. **287** (1978), 539–541.

100. ———, *CR-hypersurfaces à champ normal covariant décomposable incluses dans une variété pseudo-riemannienne neutre*, C.R. Acad. Sci. Paris Sér. I Math. **292** (1981), 287–290.

101. ———, *CR-sous-variétés co-isotropes d'une variété parakählerienne*, C.R. Acad. Sci. Paris Sér. I Math. **298** (1984), 149–151.

102. ———, *Variétés neutres M admettant une structure conforme symplectique et feuilletage coisotrope*, C.R. Acad. Sci. Paris Sér. I Math. **300** (1985), 631–634.

103. R. Rosca and L. Vanhecke, *Les sous-variétés isotropes et pseudo-isotropes d'une variété hyperbolique à n dimensions*, Verh. Konink. Acad. Wetensch. België **38** (1976), 136.

104. W. Rossmann, *The structure of semisimple symmetric spaces*, Canad. J. Math. **31** (1979), 157–180.

105. B.A. Rozenfeld, *History of non-Euclidean geometries*, Springer-Verlag, 1988.

106. ———, *On unitary and stratified spaces*, Trudy Sem. Vektor. Tenzor. Anal. **7** (1949), 260–275 (Russian).

107. B.A. Rozenfeld, *Projective geometry as metric geometry*, Trudy Sem. Vektor. Tenzor. Anal. **8** (1950), 328–354 (Russian).

108. ———, *Non-Euclidean geometries over the complex and the hypercomplex numbers and their application to real geometries*, in 125 *years of Lobatchevski non-Euclidean geometry*, Moskow-Leningrad, 1952, 151–156.

109. ———, *Non-Euclidean geometries*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moskow, 1955 (Russian).

110. B.A. Rozenfeld, N.V. Dushina and I.N. Semenova, *Differential geometry of real* 2*-surfaces in dual Hermitian Euclidean and elliptic spaces*, Trudy Geom. Sem. Kazan. Univ. **19** (1989), 107–120 (Russian).

111. H.S. Ruse, *On parallel fields of planes in a Riemannian manifold*, Quart. J. Math. Oxford Ser. **20** (1949), 218–234.

112. N. Salingaros, *Algebras with three anticommuting elements.* (II) *Two algebras over a singular field*, J. Math. Phys. **22** (1981), 2096–2100.

113. A.P. Shirokov, *On the problem of A-spaces*, in 125 *years of Lobatchevski non-Euclidean geometry*, Moskow-Leningrad, 1952 (Russian).

114. E. Study, *The geometry of dynames*, Leipzig, 1903.

115. M. Takeuchi, *Cell decompositions and Morse equalities on certain symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **12** (1965), 81–191.

116. ———, *Stability of certain minimal submanifolds of compact Hermitian symmetric spaces*, Tôhoku Math. J. **36** (1984), 293–314.

117. I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. **24** (1976), 338–351.

118. V.V. Vyshnevskij, A.P. Shirokov and V.V. Shurygin, *Spaces over algebras*, Kazan Univ., 1985 (Russian).

119. J.A. Wolf, *Spaces of constant curvature*, Publish or Perish, 1977.

120. Y.C. Wong, *Fields of parallel planes in affinely connected spaces*, Quart. J. Math. Oxford **4** (1953), 241–253.

121. I.M. Yaglom, *A simple non-Euclidean geometry and its physical basis*, Springer-Verlag, Berlin-New York, 1979.

122. K. Yano and S. Ishihara, *Tangent and cotangent bundles*, Marcel Dekker, New York, 1973.

123. Z.-Z. Zhong, *On the hyperbolic complex linear symmetry groups and their local gauge transformation actions*, J. Math. Phys. **26** (1985), 404–406.

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