Structures of electromagnetic type on vector bundles

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Abstract

Structures of electromagnetic type on a vector bundle are introduced and studied. The metric case is also defined and studied. The sets of compatible connections are determined and a canonical connection is defined.

1 Introduction

Structures of electromagnetic type (em-structures) and structures of metric electromagnetic type (mem-structures) on a manifold were progressively introduced in [9, 11, 7] (see also [6]) and studied in detail in [5, 7, 8, 13, 14]. In the present paper we define similar structures for the case of a vector bundle $\xi = (E, \pi, M)$, and relate them to product, complex, para-Hermitian, Hermitian, para-Kähler or indefinite Kähler, structures. (In the sequel, by a pseudo-Riemannian metric we shall understand a metric of any signature, and by an indefinite (metric) structure a structure including a pseudo-Riemannian metric.) Then, we determine the set of connections on ξ compatible with those structures and we introduce a canonical connection. Considering an almost para-Hermitian (resp. indefinite Hermitian) structure on the base manifold M and an indefinite Hermitian (resp. para-Hermitian) structure of the bundle ξ, we prove that the corresponding diagonal lift of these structures, with respect to a connection on ξ, are mem-structures on the total space E. Finally, some properties of those mem-structures are established.

We recall the physical origin of the topic ([9, 11]). Let M^4 be a spacetime of general relativity, with gravitational tensor g of signature $-+++$. Let F be the electromagnetic field of type $(0,2)$, which is skewsymmetric, that is a 2-form. Setting $F(X,Y) = g(JX,Y)$, the tensor field J so defined is the electromagnetic tensor field of type $(1,1)$ associated to F. We have $g(JX,Y) + g(X,JY) = 0$. The characteristic equation of J is $\det(J - \lambda I) = 0$, which is satisfied by J, and we have

$$J^4 + 2kJ^2 + lI = 0, \qquad k = -\frac{1}{4}\operatorname{trace} J^2, \quad l = \det J.$$

If $x \in M^4$, it is said that J_x is of 1^{st}, 2^{nd}, or 3^{rd} class at x if, respectively,

$$l_x \neq 0, \qquad l_x = 0,\ k_x \neq 0, \qquad l_x = 0,\ k_x = 0.$$

It is said that J is of 1^{st}, 2^{nd}, or 3^{rd} class if it is of such class at every x. The characteristic polynomial of the second class is $J^2(J^2+2k)$, but the minimal polynomial is $J(J^2+2k)$, so that the condition $J(J^2+2k)=0$ characterizes the second class. The field of an electromagnetic plane wave is of 3^{rd} class. The field of a moving electron is of 2^{nd} class. More complicated fields belong to the 1^{st} class. The equation one gets from the minimal polynomial in the 1^{st} class is

$$(J^2-f^2)(J^2+h^2)=0. \tag{1.1}$$

with f,h nowhere-vanishing C^∞ functions on M^4. Such a tensor field J on a general manifold M determines a G-structure on M.

To handle the nonconstant local cross-section situation corresponding to (1.1), one can use the relationships among G-structures, related sections of an associated bundle and functions of certain kind on M, as follows: Let $(\mathcal{P},\pi_P,M,H)$ be a principal bundle with group H, $H\times W\to W$ a left action of H on a manifold W, and $(E=\mathcal{P}\times_H W,\pi_E,M,W)$ the associated bundle. A J-subset S of W with corresponding group G, a subgroup of H, is defined by the conditions: (1) $S\subset$ fixpoint set of G, (2) $h\in H$, $h(S)\cap S\neq\emptyset\Rightarrow h\in G$. For instance, points are J-subsets with G the corresponding isotropy group. A cross-section K of π_E is a J-section if it can be locally represented as the "product" of a cross-section σ of π_P and a S-valued function $\widetilde{K}$, so that

$$K_x=\sigma_x\cdot\widetilde{K}_x=\text{equivalence class of }(\sigma_x,\widetilde{K}_x)\text{ in }E.$$

Then $\widetilde{K}$ is globally defined, and the σ generate a principal subbundle of $\mathcal{P}$. K is a constant J-section if and only if $\widetilde{K}$ is constant. Different sections can generate the same subbundle, and in fact, every principal subbundle can be generated by a constant J-section.

Now, let $\mathcal{P}$ be the principal bundle of frames over M, so that $H=GL(n,\mathbb{R})$, and let W be a real vector space. If $J\in W$ is given with the conditions stated above, a J-section generates a J-structure with group G, which is a G-structure. The tensor K has in principle *variable components* in adapted frames. This is a slight generalization with respect to the usually considered G-structures, given by tensors with constant components, which here correspond to constant J-sections. Since every J-structure is generated by some constant J-section, this generalization is useless for the study of the J-structure itself; but if the emphasis shifts to the study of variable J-sections, the results are significant, specially with respect to the parallelizability of the tensors.

In the particular case of a $(1,1)$ tensor field J satisfying $(J^2-f^2)(J^2+h^2)=0$, with characteristic polynomial $(x-p)^{r_1}(x-p)^{r_2}(x^2+q^2)^s$, $r_1,r_2,s\geq 1$, $r_1+r_2+2s=n=\dim M$, the J-subset consists of matrices of the form

$$\begin{pmatrix} pI_{r_1} & & & \\ & -pI_{r_2} & & \\ & & & -qI_s \\ & & qI_s & \end{pmatrix}$$

and the structural group is $G = GL(r_1, \mathbb{R}) \times GL(r_2, \mathbb{R}) \times GL(s, \mathbb{C})$. It is proved ([7]) that the G-structure defined by J above is also defined by a tensor field, say again J, satisfying $(J^2 - 1)(J^2 + 1) = 0$, that is, the relation $J^4 = 1$ considered in the present paper.

Notice that *the G-structure is exactly the same, not an associated or equivalent one*. In the 4-dimensional case the group reduces to $G = GL(1, \mathbb{R}) \times GL(1, \mathbb{R}) \times GL(1, \mathbb{C})$. It is also proved ([7]) that there exists an adapted Riemannian metric so that the group can be reduced to $G = O(r_1) \times O(r_2) \times U(s)$, and in the 4-dimensional case to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times U(1)$, that is, essentially to the unitary group $U(1)$.

2 Structures of electromagnetic type on a vector bundle

Let $\xi = (E, \pi, M)$ be a C^∞ vector bundle with total space E and projection map π over a connected paracompact base manifold M. The rank of E is the (common) dimension of the fibres. Let $C^\infty(M)$ denote the ring of real functions, $\mathcal{T}^p_q(M)$ the $C^\infty(M)$-module of (p, q)-tensor fields, and $\mathcal{T}(M)$ the $C^\infty(M)$-tensor algebra of M. We respectively denote by $\mathcal{T}^p_q(\xi)$ and $\mathcal{T}(\xi)$ the $C^\infty(M)$-module of tensor fields of type (p, q) and the $C^\infty(M)$-tensor algebra of the bundle ξ.

We recall that an almost product (resp. almost complex) structure on a manifold M is defined by a tensor field J of type $(1, 1)$ satisfying $J^2 = I$ (resp. $J^2 = -I$). An almost para-Hermitian (resp. indefinite almost Hermitian) structure on M is defined by a couple (J, g), given by an almost product (resp. almost complex) structure J and a pseudo-Riemannian metric compatible with J in the sense that $g(JX, Y) + g(X, JY) = 0$, $X, Y \in \mathfrak{X}(M)$; that is, as an anti-isometry (resp. isometry). A para-Kähler (resp. indefinite Kähler) manifold is a manifold M endowed with an almost para-Hermitian (resp. indefinite almost Hermitian) structure such that the Levi-Civita connection of g parallelizes J.

Definition 2.1. A *structure of electromagnetic type* on $\xi = (E, \pi, M)$ is an M-endomorphism J of ξ satisfying

$$J^4 = I,$$

with characteristic polynomial $(x - 1)^{r_1}(x + 1)^{r_2}(x^2 + 1)^s$, where r_1, r_2, s are constants greater than or equal to 1 such that $r_1 + r_2 + 2s = \operatorname{rank} E$.

Setting $P = J^2$, we have $P^2 = I$, so P is a product structure on ξ, admitting J as a “square root”. Conversely, if P is a product structure admitting a “square root” J, then J is an em-structure on ξ. Denoting by ξ_1 and ξ_2 respectively the $+1$ and -1 eigen-subbundles of P, it is easy to see that ξ_1 and ξ_2 are invariant by J and that $J_1 = J|_{\xi_1}$ defines a product structure of ξ_1 and $J_2 = J|_{\xi_2}$ a complex structure of ξ_2. So, one has

$$\xi = \xi_1 \oplus \xi_2, \qquad J = J_1 \oplus J_2. \tag{2.1}$$

Conversely, if ξ_1 and ξ_2 are two supplementary subbundles of ξ, J_1 is a product structure of ξ_1, and J_2 a complex structure of ξ_2, then $J = J_1 \oplus J_2$ is an em-structure on ξ. Denoting by P_1 and P_2 the projections of ξ on ξ_1 and ξ_2 respectively, we obtain

$$P = P_1 - P_2, \qquad J = J_1 \circ P_1 + J_2 \circ P_2.$$

Summing up we have

Proposition 2.1. *An em-structure on the vector bundle $\xi = (E, \pi, M)$ can be defined by each one of the following conditions:*
(1) *An M-endomorphism J of ξ satisfying $J^4 = I$,*
(2) *A product structure P of ξ admitting a "square root" J,*
(3) *Two supplementary subbundles ξ_1 and ξ_2 of ξ respectively endowed with a product structure and a complex structure.*

Remark 2.1. A product structure P which admits a "square root" is a particular one because rank ξ_2 must be even.

Definition 2.2. A *structure of metric electromagnetic type* (mem-structure) on the vector bundle ξ is a pair (J, g), where J is an em-structure and g a pseudo-Riemannian metric on ξ satisfying the compability condition

$$g(JX, Y) + g(X, JY) = 0, \qquad X, Y \in \xi. \tag{2.2}$$

Denoting by δ_J the derivation defined by J in the tensor algebra $\mathcal{T}(\xi)$, the relation (2.2) can be written as

$$\delta_J g = 0,$$

from which it follows $g(PX, PY) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$. Therefore, the pair (P, g) is a pseudo-Riemannian product structure of ξ and so the subbundles ξ_1 and ξ_2 are mutually orthogonal with respect to g. Denoting respectively by g_1 and g_2 the restrictions of g to ξ_1 and ξ_2, from (2.2) we obtain

$$\delta_{J_1} g_1 = 0, \qquad \delta_{J_2} g_2 = 0, \tag{2.3}$$

which may be written
(2.4)
$$g_1(J_1X, J_1X) = -g_1(X, Y), \quad g_2(J_2X, J_2Y) = g_2(X, Y), \qquad X, Y \in \mathfrak{X}(\xi).$$

Hence (J_1, g_1) is a para-Hermitian structure of ξ_1 and (J_2, g_2) is an indefinite Hermitian structure of ξ_2. Conversely, if ξ_1 and ξ_2 are two supplementary subbundles of ξ such that ξ_1 is endowed with a para-Hermitian structure (J_1, g_1) and ξ_2 with an indefinite Hermitian structure (J_2, g_2), then considering J as given by (2.1) and setting

$$g = g_1 \oplus g_2,$$

one obtains a mem-structure on ξ. So we have

Proposition 2.2. *A mem-structure (J, g) on ξ is equivalent to a pair of supplementary subbundles ξ_1 and ξ_2 respectively endowed with a para-Hermitian structure (J_1, g_1) and an indefinite Hermitian structure (J_2, g_2).*

Remark 2.2. If (J, g) is a mem-structure on ξ, then we have: $\operatorname{rank}\xi_1$ and $\operatorname{rank}\xi_2$ are even; $\operatorname{trace} J_1 = \operatorname{trace} J_2 = 0$; $\operatorname{sign} g_1 = 0$.

Setting for a mem-structure (J, g) on ξ:

$$\Omega(X, Y) = g(JX, Y), \quad \Omega_i(X, Y) = g_i(J_i X, Y), \qquad i = 1, 2,$$

it follows that Ω, Ω_1, and Ω_2 are 2-forms which determine almost symplectic structures of ξ, ξ_1 and ξ_2, so that

$$\Omega = \Omega_1 \oplus \Omega_2.$$

These 2-forms satisfy

$$\delta_J \Omega = 0, \qquad \delta_{J_1} \Omega_1 = 0, \qquad \delta_{J_2} \Omega_2 = 0. \tag{2.5}$$

Remark 2.3. The meaning of conditions (2.2), (2.3) and (2.5) is the following: The groups of automorphisms of $\mathfrak{X}(\xi_1)$, $\mathfrak{X}(\xi_2)$, and $\mathfrak{X}(\xi)$ given by

$$\alpha_t = I_1 \cosh t + J_1 \sinh t, \quad \beta_t = I_2 \cos t + J_2 \sin t, \quad \gamma_t = \alpha_t \oplus \beta_t,$$

$t \in \mathbb{R}$, determine actions on the tensor algebras $\mathcal{T}(\xi_1)$, $\mathcal{T}(\xi_2)$, and $\mathcal{T}(\xi)$, which respectively preserve the structures (J_1, g_1, Ω_1), (J_2, g_2, Ω_2), and (J, g, Ω).

3 Compatible connections

3.1 The general case

Definition 3.1. A connection D on the vector bundle ξ is said to be *compatible* with an em-structure J if

$$DJ = 0. \tag{3.1}$$

From this it follows $DP = 0$, hence D preserves the subbundles ξ_1 and ξ_2, *i.e.*, for $X \in \mathfrak{X}(M)$, $Y_1 \in \mathfrak{X}(\xi_1)$, $Y_2 \in \mathfrak{X}(\xi_2)$, one has $D_X Y_1 \in \mathfrak{X}(\xi_1)$, $D_X Y_2 \in \mathfrak{X}(\xi_2)$. Setting then

$$D^1_X Y_1 = D_X Y_1, \ D^2_X Y_2 = D_X Y_2, \qquad X \in \mathfrak{X}(M), \ Y_1 \in \mathfrak{X}(\xi_1), \ Y_2 \in \mathfrak{X}(\xi_2),$$

we have that D^1 and D^2 are respectively connections on ξ_1 and ξ_2, so that

$$D_X = D^1_X \circ P_1 + D^2_X \circ P_2, \quad D^1_X J_1 = 0, \quad D^2_X J_2 = 0, \qquad X \in \mathfrak{X}(M). \tag{3.2}$$

Conversely, if D^1 and D^2 are respectively connections on ξ_1 and ξ_2, then D given as in (3.2) is a connection on ξ satisfying $DP = 0$. If D_1 and D_2 satisfy the respective conditions in (3.2), then D satisfies (3.1) too. Thus, it follows

Proposition 3.1. *A connection D on ξ is compatible with the em-structure J if and only if there exist two connections D^1 on ξ_1 and D^2 on ξ_2, respectively compatible with the product structure J_1 and the complex structure J_2, so that*

$$D = D^1 \circ P_1 + D^2 \circ P_2. \tag{3.3}$$

Consider now on the subbundles ξ_i of ξ, the operators Φ_{J_i} and Ψ_{J_i} given by

$$(\Phi_{J_i} D^i)_X = \tfrac{1}{2}(D^i_X + J_i^{-1} \circ D^i_X \circ J_i), \ (\Psi_{J_i} \mathcal{A}^i)_X = \tfrac{1}{2}(\mathcal{A}^i_X + J_i^{-1} \circ \mathcal{A}^i_X \circ J_i), \tag{3.4}$$

where $X \in \mathfrak{X}(M)$, D^i is a connection on ξ_i, and $\mathcal{A}^i \in \Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$ (*now and in the sequel we take* $i = 1, 2$). From [1, 13] and Proposition 3.1 we obtain

Proposition 3.2. *The set of connections on ξ compatible with the em-structure J is given by*

$$D_X = \{(\Phi_{J_1} D^{\circ 1})_X + (\Psi_{J_1} \mathcal{A}^1)_X\} \circ P_1 + \{(\Phi_{J_2} D^{\circ 2})_X + (\Psi_{J_2} \mathcal{A}^2)_X\} \circ P_2,$$

where $X \in \mathfrak{X}(M)$ and $D^{\circ i}$ is an arbitrary fixed connection on ξ_i, $\mathcal{A}^i$ denotes any element of $\Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$, and Φ_{J_i}, Ψ_{J_i} are given by (3.4).

Definition 3.2. A connection D on ξ is said to be *compatible with the mem-structure* (J, g) if

$$DJ = 0, \qquad Dg = 0,$$

From which it follows: $DP = 0$; $D = D^1 \circ P_1 + D^2 \circ P_2$, where D^i are the restrictions of D to ξ_1 and ξ_2; $D^i J_i = 0$; and $D^i g_i = 0$. Conversely, if D^1 and D^2 are connections on ξ_1 and ξ_2, compatible with the para-Hermitian structure (J_1, g_1) and the indefinite Hermitian structure (J_2, g_2) respectively, then the connection D given by (3.3) is compatible with the mem-structure (J, g) on ξ. So, we have

Proposition 3.3. *A connection D on ξ is compatible with the mem-structure (J, g) on ξ, if and only if there are two connections D^1 and D^2 on the subbundles ξ_1 and ξ_2, respectively compatible with the para-Hermitian structure (J_1, g_1) and the indefinite Hermitian structure (J_2, g_2), so that D is given by* (3.3).

Setting then

$$(\Phi_{g_i} D^i)_X = \tfrac{1}{2}(D^i_X + g_i^{-1} \circ D^i_X \circ g_i), \ (\Psi_{g_i} \mathcal{A}^i)_X = \tfrac{1}{2}(\mathcal{A}^i_X + g_i^{-1} \circ \mathcal{A}^i_X \circ g_i), \tag{3.5}$$

we obtain from [1], Prop. 3.3, and (2.4)

Proposition 3.4. *The set of connections on ξ compatible with the mem-structure (J, g) is given by*

$$D_X = \big\{((\Phi_{g_1} \circ \Phi_{J_1}) D^{\circ 1})_X + ((\Psi_{g_1} \circ \Psi_{J_1}) \mathcal{A}^1)_X\big\} \circ P_1 + \big\{((\Phi_{g_2} \circ \Phi_{J_2}) D^{\circ 2})_X + ((\Psi_{g_2} \circ \Psi_{J_2}) \mathcal{A}^2)_X\big\} \circ P_2,$$

where $D^{\circ i}$ is an arbitrary fixed connection on ξ_i, $\mathcal{A}^i \in \Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$, and Φ_{J_i}, Φ_{g_i}, Ψ_{J_i}, Ψ_{g_i} are given by (3.4) *and* (3.5).

3.2 The case of the tangent bundle

We now consider the case of ξ being the tangent bundle of the manifold M, *i.e.*, $\xi = (TM, \pi, M)$. In this case, for a mem-structure (J, g) on M, the pair (P, g) is a pseudo-Riemannian almost product structure on M, and (J_1, g_1), (J_2, g_2), are respectively a para-Hermitian [4] and an indefinite Hermitian structure [10] on ξ_1 and ξ_2. If ∇ is a linear connection on M, compatible with P, *i.e.*, $\nabla P = 0$, then its restrictions ∇^1 and ∇^2 to ξ_1 and ξ_2 are connections on these subbundles. If T is the torsion tensor of ∇, we shall call *torsion tensor* of ∇^i to the tensor fields T^i given by $T^i = P_i \circ T|_{\xi_i}$, or in more detail

$$T^i(X_i, Y_i) = \nabla_{X_i} Y_i - \nabla_{Y_i} X_i - P_i[X_i, Y_i], \qquad X_i, Y_i \in \mathfrak{X}(\xi_i).$$

We call *tensors of nonholonomy* of the distributions ξ_1 and ξ_2 to the tensor fields $S^1 = P_2 \circ T|_{\xi_1}$ and $S^2 = P_1 \circ T|_{\xi_2}$, respectively. We obtain

$$S^1(X_1, Y_1) = -P_2[X_1, Y_1], \qquad S^2(X_2, Y_2) = -P_1[X_2, Y_2].$$

It follows

Proposition 3.5. *The distribution ξ_1 (resp. ξ_2) is involutive if and only if $S^1 = 0$ (resp. $S^2 = 0$).*

After some computations we obtain from [3, 10, 14]

Proposition 3.6. *For a mem-structure (J, g) on a manifold M, there exists a unique linear connection ∇ with torsion tensor T, satisfying the conditions*

$$(3.6) \qquad \nabla P = 0, \quad T(PX, Y) = T(X, PY),$$

$$(3.7) \qquad \nabla^i_{X_i} J_i = 0, \quad \nabla^i_{X_i} g_i = 0, \quad T^i(J_i X, I_i Y) = T^i(I_i X, J_i Y).$$

Definition 3.3. We shall call the *canonical connection* associated to the mem-structure (J, g) on the manifold M to the connection given by the conditions (3.6) and (3.7).

Remark 3.1. Notice that this connection slightly differs from that given in Theorem 5.3 in [14].

For the canonical connection we obtain from (3.6):

$$\nabla^1_{X_2} Y_1 = P_1[X_2, Y_1], \qquad \nabla^2_{X_1} Y_2 = P_2[X_1, Y_2].$$

Denoting by ξ_1^1, ξ_1^2 the eigen-subbundles of J_1 corresponding to $\varepsilon = +1$, $\varepsilon = -1$, by π_1^1, π_1^2 the projection maps of ξ_1 on ξ_1^1, and ξ_1^2 and by X_1^i, Y_1^i any elements of $\mathfrak{X}(\xi_1^i)$, we obtain from the first equation in (3.7)

$$\nabla^1_{X_1^2} Y_1^1 = \pi_1^1 P_1[X_1^2, Y_1^1], \qquad \nabla^1_{X_1^1} Y_1^2 = \pi_1^2 P_1[X_1^1, Y_1^2],$$

$$g_1(\nabla^1_{X_1^1} Y_1^1, Z_1^2) = X_1^1 g_1(Y_1^1, Z_1^2) - g_1([X_1^1, Z_1^2], Y_1^1),$$

$$g_1(\nabla^1_{X_1^2} Y_1^2, Z_1^1) = X_1^2 g_1(Y_1^2, Z_1^1) - g_1([X_1^2, Z_1^1], Y_1^2).$$

From the second equation in (3.7) above it results, exactly as in [14, Th. 5.1], the expression for $\nabla^2_{X_2} Y_2$.

For J and g we obtain

$$(\nabla_{X_1} J)Y_1 = 0, \quad (\nabla_{X_2} J)Y_2 = 0, \quad (\nabla_{X_1} J)Y_2 = (\nabla^2_{X_1} J_2)Y_2,$$
$$(\nabla_{X_2} J)Y_1 = (\nabla^1_{X_1} J_1)Y_1, \quad (\nabla_{X_1} g)(Y_1, Z_1) = 0, \quad (\nabla_{X_2} g)(Y_2, Z_2) = 0,$$
$$(\nabla_{X_2} g)(Y_1, Z_1) = (L_{X_2} g)(Y_1, Z_1), \quad (\nabla_{X_1} g)(Y_2, Z_2) = (L_{X_1} g)(Y_2, Z_2),$$

where L stands for the Lie derivative.

4 Structures of electromagnetic type on the total space of a vector bundle

Let $\xi = (E, \pi, M)$ be a vector bundle and (x^j), (y^a), (x^j, y^a), local coordinates in adapted charts on M, ξ, and E, respectively. We denote by (∂_j), (e_a), (∂_j, ∂_a) the corresponding local bases, where $\partial_j = \partial/\partial x^j$, $\partial_a = \partial/\partial y^a$, $j = 1, 2, \ldots, m$, $a, b, c = 1, 2, \ldots, n$ (see [2]). Setting for each $z = (x, y) \in E$, $V_z E = \operatorname{Ker} \pi_{*z}$, we obtain the *vertical distribution* and so the *vertical subbundle* of TE, denoted by VE. Let $C^{\infty v} = \{f^v = f \circ \pi : f \in C^\infty(M)\}$ be the subring of $C^\infty(E)$ naturally isomorphic to $C^\infty(M)$. Setting for each $\mu \in \Lambda^1(\xi)$, locally given by $\mu(x) = \mu_a(z) e^a$,

$$\gamma(\mu)(z) = \mu_a(x) y^a,$$

we obtain a class of functions on E enjoying the property that every vector field $A \in \mathfrak{X}(E)$ is uniquely determined by its values on those functions. The mapping γ may be extended to tensor fields $S \in \mathcal{T}^1_1(\xi)$ by

$$(\gamma S)(\gamma(\mu)) = \gamma(\mu \circ S), \qquad \mu \in \Lambda^1(\xi).$$

If $S(x) = S^a_b(x) e_a \otimes e^b$, then $\gamma S(z) = S^a_b(x) y^b \partial_a$, *i.e.*, γS is a vertical vector field on E. Now, let D be a connection on ξ and $X \in \mathfrak{X}(M)$, $u \in \mathfrak{X}(\xi)$. Setting

$$X^h(\gamma\mu) = \gamma(D_X \mu), \quad u^v(\gamma\mu) = \mu(u) \circ \pi, \qquad \mu \in \Lambda^1(\xi),$$

we obtain two vector fields X^h and u^v on E, respectively called the *horizontal lift* of X and the *vertical lift* of u. We have the useful formulas [2]:

$$(fX)^h = f^v X^h, \ (fu)^v = f^v u^v, \ [X^h, Y^h] = [X, Y]^h - \gamma R^D_{XY}, \ [u^v, w^v] = 0,$$
$$[X^h, u^v] = (D_X u)^v, \qquad f \in \mathbb{C}^\infty(M), \, X, Y \in \mathfrak{X}(M), \, u, w \in \mathfrak{X}(\xi).$$

Now, putting

$$Q(X^h) = X^h, \quad Q(u^v) = -X^v, \qquad X \in \mathfrak{X}(M), \, u \in \mathfrak{X}(\xi),$$

we obtain an almost product Q structure on E whose $+1$ and -1 eigendistributions, are respectively called the *horizontal distribution* HE of the connection D and the *vertical distribution* VE of the bundle.

For $f \in \mathcal{T}_1^1(M)$, $\varphi \in \mathcal{T}_1^1(\xi)$, $g \in \mathcal{T}_2(M)$, $\psi \in \mathcal{T}_2(\xi)$, we define the *horizontal lift* or the *vertical lift* $f^h, \varphi^v, g^h, \psi^v$, respectively by

$$
\begin{aligned}
(4.1)\quad & f^h(X^h) = f(X)^h, \quad f^h(u^v) = 0, \quad \varphi^v(X^h) = 0, \quad \varphi^v(u^v) = \varphi(u)^v, \\
& g^h(X^h, Y^h) = g(X,Y)^v, \quad g^h(X^h, u^v) = g^h(u^v, X^h) = g^h(u^v, w^v) = 0, \\
& \psi^v(X^h, Y^h) = \psi^v(X^h, u^v) = \psi^v(u^v, Y^h) = 0, \quad \psi^v(u^v, w^v) = \psi(u,w)^v, \\
& \qquad X, Y \in \mathfrak{X}(M), \ u, w \in \mathfrak{X}(\xi).
\end{aligned}
$$

We then define the *diagonal lifts* J and G for the pairs (f, φ) and (g, ψ) by

$$
(4.2) \qquad J = f^h + \varphi^v, \qquad G = g^h + \psi^v.
$$

From (4.1) and (4.2) we have

$$
J^n(X^h) = (f^n(X))^h, \quad J^n(u^v) = (\varphi^n(u))^v, \qquad n \in \mathbb{N}^*.
$$

So $J^4 = I$, that is J is an em-structure on E, if and only if $f^4 = I_1$ and $\varphi^4 = I_2$, that is, either f and φ are both em-structures or one is an em-structure and the other an almost product or almost complex structure, or finally f is an almost product (resp. almost complex) and φ is a complex (resp. product) structure on M and ξ respectively. *In the sequel we only consider the last case.*

Hence, let J be an em-structure on the total space E of ξ given by the diagonal lift in the first equation in (4.2) of an almost product (resp. almost complex) structure f on the base manifold M and a complex (resp. product) structure φ on the bundle ξ, that is, which satisfy

$$
f^2 = \varepsilon I_1, \quad \varphi^2 = -\varepsilon I_2, \qquad \varepsilon = 1 \text{ (resp. } \varepsilon = -1),
$$

with respect to a connection D on ξ. For the almost product structure P associated to J, we obtain $P = \varepsilon Q$, that is, P coincides up to the sign with the almost product structure Q above associated to D.

Now, let G be the diagonal lift in the second equation in (4.2), with respect to D, for the pair (g, ψ) of metrics on M and ξ. From (4.2) we obtain

$$
\delta_J G = (\delta_f g)^h + (\delta_\varphi \psi)^v,
$$

and so $\delta_J G = 0$ if and only if $\delta_f g = 0$ and $\delta_\varphi \psi = 0$. It follows

Proposition 4.1. *The pair (J, G) of diagonal lifts, with respect to a connection D on ξ, of an almost product* (*resp. almost complex*) *structure f on M and a complex* (*resp. product*) *structure φ of ξ, and the nondegenerate metrics g on M and ψ on ξ, is a mem-structure on the total space E of ξ if and only if the pair (f, g) is an almost para-Hermitian* (*resp. indefinite almost Hermitian*) *structure on M. The pair (φ, ψ) is an indefinite Hermitian* (*resp. para-Hermitian*) *structure on ξ.*

Denoting by ω and τ the 2-forms associated to the structures (f,g) on M and (φ,ψ) on ξ, and by Ω_1,Ω_2,Ω, the 2-forms associated to the structures (f^h,g^h) on HE, (φ^v,ψ^v) on VE and (J,G) on TE, we obtain

$$\Omega_1=\omega^h,\qquad \Omega_2=\tau^v,\qquad \Omega=\omega^h\oplus\tau^v.$$

From the hypotheses of Prop. 4.1 it follows

$$\delta_f g=0,\quad \delta_f\omega=0,\quad \delta_\varphi\psi=0,\quad \delta_\varphi\tau=0,\quad \delta_J G=0,\quad \delta_J\Omega=0.$$

Remark 4.1. The groups of automorphisms of $\mathfrak{X}(M),\mathfrak{X}(\xi),\mathfrak{X}(E)$, given respectively for $\varepsilon=1$ and $\varepsilon=-1$, by

$$\begin{aligned}
&\alpha_t=I_1\cosh t+f\sinh t,\quad \beta_t=I_2\cos t+\varphi\sin t,\quad \gamma_t=\alpha_t^h\oplus\beta_t^h,\qquad t\in\mathbb{R},\\
&\alpha_t=I_1\cos t+f\sin t,\quad \beta_t=I_2\cosh t+\varphi\sinh t,\quad \gamma_t=\alpha_t^h\oplus\beta_t^h,\qquad t\in\mathbb{R},
\end{aligned}$$

determine on the tensor algebras $\mathcal{T}(M),\mathcal{T}(\xi)$, and $\mathcal{T}(E)$, actions which preserve the structures (f,g,ω), (φ,ψ,τ) and (J,G,Ω).

For two connections ∇ on M and D on ξ, we define the *horizontal lift* ∇^h on the subbundle HE and the *vertical lift* D^v on the subbundle VE (each one with respect to the connection D), respectively by

$$\nabla^h_{X^h}Y^h=(\nabla_XY)^h,\quad \nabla^h_{u^v}Y^h=0,\quad D^v_{X^h}w^v=(D_Xw)^v,\quad D^v_{u^v}w^v=0.$$

Putting them

$$\mathcal{D}_AX=\nabla^h_A HX+D^v_A VX,\quad A,X\in\mathfrak{X}(E),$$

where H and V denote the horizontal and vertical projectors of TE on HE and VE, we obtain a linear connection $\mathcal{D}$ on E, called the *diagonal lift* of the pair (∇,D) with respect to the connection D (see [2]), whose restrictions to the subbundles $\xi_1=HE$ and $\xi_2=VE$ are $\mathcal{D}_1=\nabla^h$ and $\mathcal{D}_2=D^v$. The nonvanishing components of the torsion and curvature tensors of $\mathcal{D}$ are given by

$$\begin{aligned}
&\mathcal{T}(X^h,Y^h)=T^\nabla(X,Y)^h+\gamma R^D_{XY},\\
&\mathcal{R}_{X^hY^h}Z^h=(R^\nabla_{XY}Z)^h,\quad \mathcal{R}_{X^hY^h}u^v=(R^D_{XY}u)^v,
\end{aligned}\tag{4.3}$$

where T^∇,R^∇, and R^D stand for the torsion tensor of ∇ and the curvature tensors of ∇ and D.

For the covariant derivatives, with respect to $\mathcal{D}$, of the horizontal lift of f and g, and the vertical lift of φ and ψ we obtain

$$\begin{aligned}
&\mathcal{D}_{X^h}f^h=(\nabla_Xf)^h,\quad \mathcal{D}_{u^v}f^h=0,\quad \mathcal{D}_{X^h}g^h=(\nabla_Xg)^h,\quad \mathcal{D}_{u^v}g^h=0,\\
&\mathcal{D}_{X^h}\varphi^v=(D_X\varphi)^v,\quad \mathcal{D}_{u^v}\varphi^v=0,\quad \mathcal{D}_{X^h}\psi^v=(D_X\psi)^v,\quad \mathcal{D}_{u^v}\psi^v=0.
\end{aligned}$$

So, for the diagonal lifts J and G of the pairs (f,φ) and (g,ψ), it follows

$$\begin{aligned}(4.4)\qquad \mathcal{D}_{X^h}J &= (\nabla_X f)^h + (D_X\varphi)^v, \qquad \mathcal{D}_{u^v}J = 0,\\ \mathcal{D}_{X^h}G &= (\nabla_X g)^h + (D_X\psi)^v, \qquad \mathcal{D}_{u^v}G = 0.\end{aligned}$$

Hence, $\mathcal{D}J = 0$ if and only if $\nabla f = 0$, $D\varphi = 0$; and $\mathcal{D}G = 0$ if and only if $\nabla g = 0$, $D\psi = 0$. From (4.3) and (4.4) it follows, for $P = J^2$, that $\mathcal{D}P = 0$ and $\mathcal{T}\circ P\times I = \mathcal{T}\circ I\times P$ for any connections ∇ on M and D on ξ. After that we have

$$\nabla^h_{X^h}g^h = (\nabla_X g)^h, \quad D^v_{u^v}\varphi^v = 0, \quad D^v_{u^v}\psi^v = 0,$$
$$\nabla^h_{X^h}f^h = (\nabla_X f)^h, \quad \mathcal{T}^1(f^hX, I_1Y) = (T^\nabla(fX, I_1Y))^h, \quad \mathcal{T}^2(\varphi^vX, I_2Y) = 0,$$

where $\mathcal{T}^1 = H\circ\mathcal{T}|_{HE}$ and $\mathcal{T}^2 = V\circ\mathcal{T}|_{VE}$. So we obtain

Proposition 4.2. *The diagonal lift $\mathcal{D}$ on E, for the connections ∇ on M and D on ξ, is the canonical connection associated to the mem-structure (J,G) if and only if*

$$\nabla f = 0, \quad \nabla g = 0, \quad T^\nabla(fX,Y) = T^\nabla(X,fY),$$

i.e., *the connection ∇ is the canonical connection* [2, 10] *associated to the almost para-Hermitian (resp. indefinite almost Hermitian) structure (f,g) on M.*

Also from (4.3) and (4.4) we obtain $\mathcal{D}G = 0$ and $\mathcal{T} = 0$ if and only if $\nabla g = 0$, $T^\nabla = 0$, $R^D = 0$ and $D\psi = 0$. Hence we have

Proposition 4.3. *The diagonal lift $\mathcal{D}$ of the pair of connections (∇, D) coincides with the Levi-Civita connection of G if and only if ∇ is the Levi-Civita connection of g, D has vanishing curvature and ψ is covariant constant.*

For the Nijenhuis tensor of J,

$$N_J(A,B) = [JA,JB] + J^2[A,B] - J[JA,B] - J[A,JB], \qquad A,B\in\mathfrak{X}(E),$$

we obtain

$$\begin{aligned}(4.5)\quad N_J(X^h,Y^h) &= N_f(X,Y)^h + \gamma\big(\varepsilon R^D_{XY} - R^D_{fXfY} + \varphi\circ(R^D_{fXY} + R^D_{XfY})\big),\\ N_J(X^h,u^v) &= \big(D_{fX}\varphi u - \varepsilon D_X u - \varphi\circ(D_{fX}u + D_X\varphi u)\big)^v, \quad N_J(u^v,w^v) = 0.\end{aligned}$$

It follows

Proposition 4.4. *The mem-structure J is integrable* (i.e., *$N_J = 0$, see* [8]) *if and only if f is a product (resp. a complex) structure in M, the connection D has vanishing curvature and the complex (resp. product) structure φ on ξ is covariant constant.*

For the exterior differential of the 2-form Ω associated to the mem-structure (J,G) we obtain

$$d\Omega(X^h,Y^h,Z^h) = d\omega(X,Y,Z)^v, \qquad 3d\Omega(X^h,Y^h,w^v) = -\gamma(i_w\tau\circ R^D_{XY}),$$
$$3d\Omega(X^h,u^v,w^v) = D_X\tau(u,w)^v, \qquad d\Omega(u^v,v^v,w^v) = 0.$$

Hence

Proposition 4.5. *The almost symplectic structure Ω associated to the mem-structure (J, G) on E is integrable* (i.e., $d\Omega = 0$) *if and only if the structure (f, g) is almost para-Kähler (resp. indefinite almost Kähler), the connection D has vanishing curvature, and the 2-form τ on ξ is covariant constant.*

Finally we obtain

Proposition 4.6. *For the mem-structure (J, G) on E, the structures J and Ω are simultaneously integrable if and only if the structure (f, g) is a para-Kähler (resp. indefinite Kähler) structure on M, D has vanishing curvature and the pair (φ, ψ) is covariant constant.*

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