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ALMOST PRODUCT BICOMPLEX STRUCTURES ON MANIFOLDS*

BY

V. CRUCEANU

Abstract. We study the equivalence of an almost product bicomplex (apbc)-structure with other important structures on a manifold, metrics and linear connections compatible with such a structure and the integrability of the apbc-structures. Finally, we give an example of an apbc-structure on the tangent bundle of an almost Hermitian manifold.

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1. Introduction. The almost product bicomplex (apbc)-structures, together with other important structures on a manifold, were considered by Libermann [9], Hsu [7], Cruceanu [3], Maksym and Zmurek [10] and others. But a more complete and consistent analyze of these structures was made by Bonome, Castro, Garcia-Rio, Hervella and Matsushita in the joint paper [2].

In this work we study the equivalence of an apbc-structure with other important structures on a manifold, metrics and linear connections compatible with such a structure and the integrability of the metric apbc-structures. An example of a Riemannian apbc-structure on the total space of the tangent bundle to an almost Hermitian manifold is also analyzed.

2. Almost product bicomplex structures. Let M be a paracompact and connected C^∞-manifold, $\mathcal{F}(M)$ the ring of real functions, $\mathcal{D}_s^r(M)$

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the $\mathcal{F}(M)$-module of (r,s)-tensor fields and $\mathcal{D}(M)$ the $\mathcal{F}(M)$-tensor algebra on M.

Definition 2.1. An almost product bicomplex (apbc)-structure on the manifold M, is a triple (F,G,H) of $(1,1)$-tensor fields which satisfies the conditions

$$-F^2 = G^2 = H^2 = F \circ G \circ H = -I, F \neq \pm I. \tag{2.1}$$

It follows that F is an almost product (ap)-structure and G, H are almost complex (ac)-structures on M, which satisfy the relations

$$F \circ G = G \circ F = H, G \circ H = H \circ G = -F, H \circ F = F \circ H = G, F \neq \pm I. \tag{2.2}$$

Denote by $V_1 = F^+$ and $V_2 = F^-$, the eigendistributions (or subbundles of TM), corresponding to eigenvalues ± 1 and by F_1 and F_2 the associated projectors to F, i.e.

$$F_1 = \frac{I+F}{2}, \quad F_2 = \frac{I-F}{2}. \tag{2.3}$$

Setting then

$$\varphi_1 = G \circ F_1, \quad \varphi_2 = G \circ F_2, \tag{2.4}$$

one obtains

$$\begin{aligned} &\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0,\ \varphi_1^2 = -F_1,\ \varphi_2^2 = -F_2, \\ &\varphi_1^2 + \varphi_2^2 = -I,\ \varphi_1^3 + \varphi_1 = \varphi_2^3 + \varphi_2 = 0. \end{aligned} \tag{2.5}$$

Definition 2.2. An almost cocomplex (acc)-structure on M is a $(1,1)$-tensor field φ satisfying $\varphi^3 + \varphi = 0$. Two (acc)-structures φ_1 and φ_2 are supplementary if $\varphi_1^2 + \varphi_2^2 = -I$.

From (2.4) and (2.5) we obtain

$$F = \varphi_2^2 - \varphi_1^2,\ \ G = \varphi_1 + \varphi_2,\ \ H = \varphi_1 - \varphi_2. \tag{2.6}$$

Then, from (2.2) it follows that G and H preserve the distributions V_1 and V_2 and so, putting $\varphi_1' = G/V_1, \varphi_2' = G/V_2$, one has ${\varphi_1'}^2 = -I_1, {\varphi_2'}^2 = -I_2$, i.e. φ_1' and φ_2' are complex structures on V_1 and V_2 respectively.

Definition 2.3. An almost CR-structure [1] on a manifold M is a pair (D, J), where D is a distribution on M and J an almost complex structure on D. Two almost CR-structures (D_1, J_1) and (D_2, J_2) are supplementary if D_1 and D_2 are supplementary distributions on M.

It follows that (V_1, φ'_1) and (V_1, φ'_2) are supplementary almost CR-structures on M and from (2.4) one has

$$G = \varphi'_1 \circ F_1 + \varphi'_2 \circ F_2, \quad H = \varphi'_1 \circ F_1 - \varphi'_2 \circ F_2. \tag{2.7}$$

From the previous considerations it results.

Theorem 2.1. *An apbc-structure on the manifold M may be defined by one of the following equivalent structures:*

1) *A triple formed by an ap-structure F and two ac-structures G and H which satisfy $F \circ G \circ H = -I, F \neq \pm I$.*

2) *A pair formed by an ap-structure F and an ac-structure G (or H), which commute.*

3) *Two commuting ac-structures, G and H, with $G \neq \pm H$.*

4) *Two supplementary acc-structures φ_1 and φ_2, with $\varphi_1 \neq 0, I$.*

5) *Two supplementary almost CR-structures (V_1, φ'_1) and (V_2, φ'_2).*

V_1 and V_2 being complex distributions, it results $\dim V_1 = 2n_1, \dim V_2 = 2n_2$ and so $\dim M = 2(n_1+n_2)$. In particular, if F is an almost paracomplex (apc)-structure [5], on M, i.e. $F^2 = I, TrF = 0$, then $n_1 = n_2 = n$ and hence $\dim M = 4n$.

Definition 2.4. An adapted basis, for an apbc-structure (F, G, H) in $x \in M$, is a basis $(e_i, e_{n_1+i}, e_a, e_{n_2+a})$, with $e_i \in V_1,\ e_{n_1+i} = G(e_i), e_a \in V_2, e_{n_2+a} = G(e_a), i = 1, 2, \ldots, n_1, a = 1, 2, \ldots, n_2$.

In an adapted basis, the tensor fields F, G, H, φ_1 and φ_2 have the matrices

$$\begin{aligned} &F = \begin{bmatrix} I_{2n_1} & 0 \\ 0 & -I_{2n_2} \end{bmatrix}, G = \begin{bmatrix} \varphi'_1 & 0 \\ 0 & \varphi'_2 \end{bmatrix}, H = \begin{bmatrix} \varphi'_1 & 0 \\ 0 & -\varphi'_2 \end{bmatrix}, \\ &\varphi_1 = \begin{bmatrix} \varphi'_1 & 0 \\ 0 & 0 \end{bmatrix}, \varphi_2 = \begin{bmatrix} 0 & 0 \\ 0 & \varphi'_2 \end{bmatrix} \end{aligned} \tag{2.8}$$

with

$$\varphi_1' = \begin{bmatrix} 0 & -I_{n_1} \\ I_{n_1} & 0 \end{bmatrix}, \quad \varphi_2' = \begin{bmatrix} 0 & -I_{n_2} \\ I_{n_2} & 0 \end{bmatrix}. \tag{2.9}$$

The change of the adapted bases are given by matrices of the form

$$T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \text{ with } A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, B = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} \tag{2.10}$$

and $a + ib \in GL(n_1, \mathbb{C}), p + iq \in GL(n_2, \mathbb{C})$.

It follows from here.

Theorem 2.2. *The structural group of the tangent bundle of a manifold M endowed with an apbc-structure is reducible to the real representation $\Sigma(2n_1, \mathbb{R}) \times \Sigma(2n_2, \mathbb{R})$ of the direct product $GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C})$.*

3. Metric and symplectic structures compatible with an apbc-structure. Let h be a metric structure on M and

$$\begin{aligned} &g_1 = h \circ (I \times I + F \times F + G \times G + H \times H), \quad g_2 = g_1 \circ I \times F, \\ &\omega_1 = g_1 \circ I \times G, \quad \omega_2 = g_1 \circ I \times H. \end{aligned} \tag{3.1}$$

One obtains

$$\begin{aligned} &g_\alpha \circ F \times F = g_\alpha \circ G \times G = g_\alpha \circ H \times H = g_\alpha, \\ &\omega_\alpha \circ F \times F = \omega_\alpha \circ G \times G = \omega_\alpha \circ H \times H = \omega_\alpha, \quad \alpha = 1, 2, \end{aligned} \tag{3.2}$$

i.e. g_α are metric and ω_α are almost symplectic structures on M, compatible with the apbc-structure (F, G, H).

In particular, if h is a Riemannian structure then g_1 is also Riemannian and g_2 is pseudo-Riemannian structure of signature (n_1, n_2).

Denoting then

$$\gamma_1' = g_1 / V_1 \times V_1, \quad \gamma_2' = g_2 / V_2 \times V_2, \tag{3.3}$$

we obtain two metrics γ_1' on V_1 and γ_2' on V_2, which are Riemannian in the same time with g_1 and satisfy

$$\gamma_\alpha' \circ \varphi_\alpha' \times \varphi_\alpha' = \gamma_\alpha', \quad \alpha = 1, 2. \tag{3.4}$$

Considering

$$\psi_1 = \omega_1 \circ F_1 \times F_1, \ \ \psi_2 = \omega_2 \circ F_2 \times F_2, \tag{3.5}$$

we obtain two degenerate 2-forms on M and we have

$$\omega_1 = \psi_1 + \psi_2, \ \ \omega_2 = \psi_1 - \psi_2. \tag{3.6}$$

After that, setting:

$$\psi_1' = \omega_1 / V_1 \times V_1, \quad \psi_2' = \omega_2 / V_2 \times V_2, \tag{3.7}$$

one obtains two symplectic forms on V_1 and V_2, which satisfy

$$\psi_\alpha' \circ \varphi_\alpha' \times \varphi_\alpha' = \psi_\alpha', \quad \alpha = 1, 2. \tag{3.8}$$

Definition 3.1. We call the set (F, G, H, g_1), which satisfy (2.1) and (3.1), a metric almost product bicomplex (mapbc)-structure on M and g_2, ω_1, ω_2 the associated metric and almost symplectic structures.

Therefore, to a Riemannian mapbc-structure (F, G, H, g_1) we will associate the follows structures: the Riemannian ap-structure (F, g_1) with the associated pseudo Riemannian structure g_2, the pseudo-Riemannian ap-structure (F, g_2) with the associated Riemannian structure g_1, the almost Hermitian structures (G, g_1) and (H, g_1) with the associated almost symplectic structures ω_1 and ω_2 respectively, and the indefinit almost Hermitian structures (G, g_2) and (H, g_2) with the associated almost symplectic structures ω_2 and ω_1 respectively. We will have also, on the distributions V_α, the Hermitian structures $(\varphi_\alpha', \gamma_\alpha')$ with the associated symplectic structures $\psi_\alpha', \alpha = 1, 2$.

Definition 3.2. An adapted basis for the Riemannian mapbc-structure (F, G, H, g_1) is an adapted basis for the abpc-structure (F, G, H), which is orthonormal with respect to g_1.

In such a basis the matrices of g_1, g_2, ω_1 and ω_2 coincide with the matrices of I, F, G, H respectively. From here and the Theorem 2.2, it follows

Theorem 3.1. *The structural group of the tangent bundle for a manifold M endowed with a Riemannian mapbc-structure is reducible to the real representation $SO(2n_1) \times SO(2n_2)$ of the direct product $\mathcal{U}(n_1, \mathbb{C}) \times \mathcal{U}(n_2, \mathbb{C})$.*

4. Connections compatible with an apbc-structure. For to give a more geometrical character to our considerations, we will use from the beginning the following important remark. If ∇^0 is a fixed connection on M, then for each connection ∇ there exists a single tensor field $\tau \in \mathcal{D}^1_2(M)$ so that $\nabla = \nabla^0 + \tau$. With other words, the set $C(M)$ of linear connections on M is an $\mathcal{F}(M)$-affine module [4], associated to the $\mathcal{F}(M)$-linear module $\mathcal{D}^1_2(M)$.

Considering now an ap-structure F on M and setting for $\nabla \in C(M), \tau \in \mathcal{D}^1_2(M), X \in \mathcal{D}^1(M)$,

$$\psi_F(\nabla)_X = \frac{1}{2}(\nabla_X + F \circ \nabla_X \circ F), \chi_F(\tau)_X = \frac{1}{2}(\tau_X + F \circ \tau_X \circ F), \tag{4.1}$$

we get that $\psi_F(\nabla) \in \mathcal{C}(M), \chi_F(\tau) \in \mathcal{D}^1_2(M)$ and

$$\psi_F^2 = \psi_F, \ \ \chi_F^2 = \chi_F, \ \ \psi_F(\nabla + \tau) = \psi_F(\nabla) + \chi_F(\tau). \tag{4.2}$$

It follows from here that ψ_F is the $\mathcal{F}(M)$-affine projector on $C(M)$ associated to the $\mathcal{F}(M)$-linear projector χ_F on $\mathcal{D}^1_2(M)$.

Definition 4.1. A linear connection ∇ on M is called compatible with the ap-structure F,or is a F-connection, if $\nabla F = 0$.

From (4.1) and (4.2) it follows that $\nabla F = 0$ is equivalent with $\psi_F(\nabla) = \nabla$ and so with $C_F(M) = Im\psi_F$. Hence we have

Theorem 4.1. *The set $C_F(M)$ of connection on M, compatible with the ap-structure F, is the affine submodule of $C(M)$ which coincides with the image of the affine projector ψ_F.*

Considering on $C(M)$ the conjugation with respect to F, i.e. the automorphism $\kappa_F : C(M) \to C(M)$ given by

$$\kappa_F(\nabla)_X = F \circ \nabla_X \circ F, \ \ \forall \nabla \in C(M), X \in \mathcal{D}^1(M), \tag{4.3}$$

we obtain

$$\psi_F(\nabla) = \frac{1}{2}(\nabla + \kappa_F(\nabla)). \tag{4.4}$$

Hence κ_F is the affine symmetry of the affine module $C(M)$, with respect to affine submodule $C_F(M)$, made parallel with the linear submodule Ker χ_F and ψ_F is the mean connection of ∇ and its conjugate $\chi_F(\nabla)$, with respect to F. We will call $\psi_F(\nabla)$ the *F-connection associated to* ∇, with respect to ap-structure F. Using the projectors F_1 and F_2, the F-connection $\psi_F(\nabla)$ may be also given by

$$\psi_F(\nabla)_X = \sum_{\alpha=1}^{2} F_\alpha \circ \nabla_X \circ F_\alpha, \quad X \in \mathcal{D}^1(M). \tag{4.5}$$

But being a F-connection, $\psi_F(\nabla)$ preserves the subbundles V_1, V_2 and induces on them the connections

$$\overset{\alpha}{\nabla}_X Y_\alpha = F_\alpha \circ \nabla_X Y_\alpha, \quad X \in \mathcal{D}^1(M), Y_\alpha \in V_\alpha, \alpha = 1, 2. \tag{4.6}$$

and so we have

$$\psi_F(\nabla)_X = \sum_{\alpha=1}^{2} \overset{\alpha}{\nabla}_X \circ F_\alpha. \tag{4.7}$$

Let ∇^0 be a fixed connection on M. Since $C_F(M) = Im\psi_F$ then, for each connection $\nabla \in C_F(M)$, there exists $\nabla' \in C(M)$ so that $\nabla = \psi_F(\nabla')$. After that, there exists $\tau \in \mathcal{D}_2^1(M)$ so that $\nabla' = \nabla^0 + \tau$. Therefore, $\nabla = \psi_F(\nabla^0 + \tau)$ and from (4.2) it results

Theorem 4.2. *The set $C_F(M)$ of connections ∇ on M compatible with the ap-structure F is given by*

$$\nabla = \psi_F(\nabla^0) + \chi_F(\tau), \tag{4.8}$$

where ∇ is a fixed connection and τ an arbitrary (1.2)-tensor field on M.

With other words, $C_F(M)$ is the affine submodule of $C(M)$ passing through the F-connection $\psi_F(\nabla^0)$ and having the direction given by the linear submodule $Im\chi_F$ of $\mathcal{D}_2^1(M)$. Similarly considering an ac-structure G on M and setting for $\nabla \in C(M), \tau \in \mathcal{D}_2^1(M)$ and $X \in \mathcal{D}^1(M)$,

$$\psi_G(\nabla)_X = \frac{1}{2}(\nabla_X - G \circ \nabla_X \circ G), \quad \chi_G(\tau)_X = \frac{1}{2}(\tau_X - G \circ \tau_X \circ G), \tag{4.9}$$

we obtain

$$\psi_G^2 = \psi_G, \quad \chi_G^2 = \chi_G, \quad \psi_G(\nabla + \tau) = \psi_G(\nabla) + \chi_G(\tau). \tag{4.10}$$

After that, for the set of G-connections and the conjunction with respect to G, we have $C_G(M) = Im\psi_G, \kappa_G(\nabla)_X = -G \circ \nabla_X \circ G$ and so,

$$\psi_G(\nabla) = \frac{1}{2}(\nabla + \kappa_G(\nabla)). \tag{4.11}$$

Finally, the affine submodule $C_G(M)$ of G-connections is given by

$$\nabla = \psi_G(\nabla^0) + \chi_G(\tau), \tag{4.12}$$

with fixed $\nabla^0 \in C(M)$ and arbitrary $\tau \in \mathcal{D}^1_2(M)$.

Definition 4.2. A connection ∇ is called compatible with the apbc-structure (F, G, H) or is a (F, G, H)-connection on M, if it satisfies

$$\nabla F = \nabla G = \nabla H = 0. \tag{4.13}$$

By an easy calculation we obtain

Theorem 4.3. *A connection ∇ on M is a (F, G, H)-connection iff it satisfies one of the following conditions:*

1. *∇ is a (F, G) or a (G, H) or a (H, F)-connection,*
2. *∇ is a (φ_1, φ_2)-connection,*
3. *There exist a φ'_1-connection $\overset{1}{\nabla}$ on V_1 and a φ'_2-connection $\overset{2}{\nabla}$ on V_2 so that*

$$\nabla_X = \overset{1}{\nabla}_X \circ F_1 + \overset{2}{\nabla}_X \circ F_2, \quad \forall X \in \mathcal{D}^1(M). \tag{4.14}$$

From the commutativity of the composition for F, G, H it follows the commutativity for the composition of ψ_F, ψ_G, ψ_H; of χ_F, χ_G, χ_H and of $\kappa_F, \kappa_G, \kappa_H$. After that ψ_F and ψ_G being affine projectors associated to linear projectors χ_F and χ_G it results that $\psi_F \circ \psi_G$ is the affine projector associated to linear projector $\chi_F \circ \chi_G$, i.e.

$$\psi_F \circ \psi_G(\nabla + \tau) = \psi_F \circ \psi_G(\nabla) + \chi_F \circ \chi_G(\tau) \tag{4.15}$$

From here one obtains

Theorem 4.4. *The set $C_{FGH}(M)$ of connections compatible with the apbc-structure (F,G,H) is given by*

$$\nabla = \psi_F \circ \psi_G(\nabla^0) + \chi_F \circ \chi_G(\tau), \tag{4.16}$$

with $\nabla^0 \in C(M)$ fixed and $\tau \in \mathcal{D}_2^1(M)$ arbitrary.

Taking here $\tau = 0$, it follows that an apbc-structure (F,G,H) assign to each connection $\nabla^0 \in C(M)$ a (F,G,H)-connection $\nabla = \psi_F \circ \psi_G(\nabla^0)$ which may be written also in the form

$$\nabla = \frac{1}{4}(\nabla^0 + \kappa_F(\nabla^0) + \kappa_G(\nabla^0) + \kappa_H(\nabla^0)), \tag{4.17}$$

i.e. ∇ is the mean connection of ∇^0 and its conjugate connections with respect to F, G and H.

Now let g be a metric on M, considered as a mapping from $\mathcal{D}^1(M)$ to $\mathcal{D}_1(M)$ which assigns to a vector field X the 1-form α given by $\alpha(Y) = g(X,Y)$, for any vector field Y. Setting then, for $\nabla \in C(M)$ and $\tau \in \mathcal{D}_2^1(M)$,

$$\psi_g(\nabla)_X = \frac{1}{2}(\nabla_X + g^{-1} \circ \nabla_X \circ g), \quad \chi_g(\tau)_X = \frac{1}{2}(\tau_X + g^{-1} \circ \tau_X \circ g), \tag{4.18}$$

we obtain as for an ap-structure F, the following

Theorem 4.5. *The set $C_g(M)$ of connections on M compatible with a metric g (i.e. $\nabla g = 0$) are given by*

$$\nabla = \psi_g(\nabla^0) + \chi_g(\tau), \tag{4.19}$$

with fixed $\nabla^0 \in C(M)$ and arbitrary $\tau \in \mathcal{D}_2^1(M)$.

Definition 4.3. A connection ∇ is called compatible with a mapbc-structure (F,G,H,g), or is a (F,G,H,g)-connection on M, if it satisfies

$$\nabla F = \nabla G = \nabla H = \nabla g = 0. \tag{4.20}$$

For a (F,G,H,g_1)-connection ∇ on M we have also

$$\begin{aligned} &\nabla F_\alpha = \nabla \varphi_\alpha = \nabla g_2 = \nabla \omega_\alpha = \nabla \psi_\alpha = 0, \\ &\overset{\alpha}{\nabla} \varphi'_\alpha = \overset{\alpha}{\nabla} \gamma'_\alpha = \overset{\alpha}{\nabla} \psi'_\alpha = 0, \quad \alpha = 1,2. \end{aligned} \tag{4.21}$$

Using (2.2), we obtain for a mapbc-structure (F, G, H, g_1), $\psi_{g_1} \circ \psi_F = \psi_F \circ \psi_{g_1}$, etc. $\chi_{g_1} \circ \chi_F = \chi_F \circ \chi_{g_1}$, etc and so for the connections compatible with such a structure one obtains

Theorem 4.6. *The set $C_{FGHg_1}(M)$ of connections on M, compatible with the mapbc-structure (F, G, H, g_1) is given by*

$$\nabla = \psi_F \circ \psi_G \circ \psi_{g_1}(\nabla^0) + \chi_F \circ \chi_G \circ \chi_{g_1}(\tau), \tag{4.22}$$

with fixed $\nabla^0 \in C(M)$ and arbitrary $\tau \in \mathcal{D}_2^1(M)$.

In particular, taking here $\tau = 0$ and $\nabla^0 = \nabla^{g_1}$ or $\nabla^0 = \nabla^{g_2}$, i.e. the Levi-Civita connections of the metrics g_1 and g_2, we obtain

Theorem 4.7. *The connections $D^\alpha = \psi_F \circ \psi_G(\nabla^{g_\alpha}), \alpha = 1, 2$, associated to Levi-Civita connections of g_α, are compatible with the mapbc-structure (F, G, H, g_1).*

5. Integrability for the apbc-structure (F, G, H).

Considering the Nijenhus tensor for $\varphi_1, \varphi_2, (\varphi_1, \varphi_2), F, G, H$ and taking $X_\alpha \in V_\alpha, \alpha = 1, 2$ we obtain

$$N_{\varphi_1}(X_1, Y_1) = [\varphi_1 X_1, \varphi_1 Y_1] + \varphi_1^2[X_1, Y_1] - \varphi_1[\varphi_1 X_1, Y_1] - \varphi_1[X_1, \varphi_1 Y_1],$$
$$N_{\varphi_1}(X_1, Y_2) = \varphi_1(\varphi_1[X_1, Y_2] - [\varphi_1 X_1, Y_2]),$$
$$N_{\varphi_1}(X_2, Y_2) = \varphi_1^2[X_2, Y_2] = -F_1[X_2, Y_2],$$
$$N_{\varphi_2}(X_1, Y_1) = \varphi_2^2[X_1, Y_1] = -F_2[X_1, Y_1],$$
$$N_{\varphi_2}(X_1, Y_2) = \varphi_2(\varphi_2[X_1, Y_2] - [X_1, \varphi_2 Y_2]),$$
$$N_{\varphi_2}(X_2, Y_2) = [\varphi_2 X_2, \varphi_2 Y_2] + \varphi_2^2[X_2, Y_2] - \varphi_2[\varphi_2 X_2, Y_2] - \varphi_2[X_2, \varphi_2 Y_2],$$
$$N_{\varphi_1\varphi_2}(X_1, Y_1) = -\varphi_2([\varphi_1 X_1, Y_1] + [X_1, \varphi_1 Y_1]),$$
$$N_{\varphi_1\varphi_2}(X_2, Y_2) = -\varphi_1([\varphi_2 X_2, Y_2] + [X_2, \varphi_2 Y_2]),$$
$$N_{\varphi_1\varphi_2}(X_1, Y_2) = [\varphi_1 X_1, \varphi_2 Y_2] - \varphi_1[X_1, \varphi_2 Y_2] - \varphi_2[\varphi_1 X_1, Y_2].$$
$$N_F(X_1, Y_1) = 4F_2[X_1, Y_1], \quad N_F(X_1, Y_2) = 0, \quad N_F(X_2, Y_2) = 4F_1[X_2, Y_2].$$
$$N_G(X_\alpha, Y_\alpha) = (N_{\varphi_1} + N_{\varphi_2} + N_{\varphi_1\varphi_2})(X_\alpha, Y_\alpha), \alpha = 1, 2,$$
$$N_G(X_1, Y_2) = (N_{\varphi_1\varphi_2} - N_{\varphi_1\varphi_2} \circ \varphi_1 \times \varphi_2)(X_1, Y_2).$$
$$N_H(X_\alpha, Y_\alpha) = (N_{\varphi_1} + N_{\varphi_2} - N_{\varphi_1\varphi_2})(X_\alpha, Y_\alpha), \quad \alpha = 1, 2,$$
$$N_H(X_1, Y_2) = -(N_{\varphi_1\varphi_2} + N_{\varphi_1\varphi_2} \circ \varphi_1 \times \varphi_2)(X_1, Y_2).$$

From these formulas it results:

Theorem 5.1. 1. The distribution V_1 is involutive iff one of the following conditions is satisfied;
(5.2)
$$N_F(X_1,Y_1)=0;\ \ F_2[X_1,Y_1]=0;\ \ N\varphi_2(X_1,Y_1)=0;\ \ \varphi_2[X_1,Y_1]=0.$$

2. The distribution V_2 is involutive iff one of the following conditions is satisfied:
(5.3)
$$N_F(X_2,Y_2)=0;\ \ F_1[X_2,Y_2]=0;\ \ N\varphi_1(X_2,Y_2)=0;\ \ \varphi_1[X_2,Y_2]=0.$$

3. Both V_1 and V_2 are involutive iff one of the following conditions is satisfied

$$\begin{aligned} &N_F=0;\ \ F_2[X_1,Y_1]=F_1[X_2,Y_2]=0;\\ &N\varphi_2(X_1,Y_1)=N\varphi_1(X_2,Y_2)=0;\ \ \varphi_2[X_1,Y_1]=\varphi_1[X_2,Y_2]=0. \end{aligned} \tag{5.4}$$

4. The ac-structure G is integrable iff $N_G=0$ or

$$\begin{aligned} &(N\varphi_1+N\varphi_2+N\varphi_1\varphi_2)(X_\alpha,Y_\alpha)=0,\alpha=1,2,\\ &(N_{\varphi_1\varphi_2}-N_{\varphi_1\varphi_2}\circ\varphi_1\times\varphi_2)(X_1,Y_2)=0. \end{aligned} \tag{5.5}$$

5. The ac-structure H is integrable iff $N_H=0$ or

$$\begin{aligned} &(N_{\varphi_1}+N\varphi_2-N_{\varphi_1\varphi_2})(X_\alpha,Y_\alpha)=0,\alpha=1.2,\\ &(N\varphi_1\varphi_2+N_{\varphi_1\varphi_2}\circ\varphi_1\times\varphi_2)(X_1,Y_2)=0. \end{aligned} \tag{5.6}$$

6. Both G and H are integrable iff $N_G=N_H=0$ or

$$(N_{\varphi_1}+N_{\varphi_2})(X_\alpha,Y_\alpha)=0,\alpha=1,2,\ \ N_{\varphi_1\varphi_2}=0. \tag{5.7}$$

7. If $N_F=0$, then $N_{\varphi_1}(X_2,Y_2)=N_{\varphi_2}(X_1,Y_1)=0,N_{\varphi_1\varphi_2}(X_\alpha,Y_\alpha)=0,\alpha=1,2$ and in this hypothesis one has.

7. a) G is integrable iff $N_{\varphi_\alpha}(X_\alpha,Y_\alpha)=0,\alpha=1,2$,
$(N_{\varphi_1\varphi_2}-N_{\varphi_1\varphi_2}\circ\varphi_1\times\varphi_2)(X_1,Y_2)=0$.

7. b) H is integrable iff $N\varphi_\alpha(X_\alpha,Y_\alpha)=0,\alpha=1,2$,
$(N_{\varphi_1\varphi_2}+N_{\varphi_1\varphi_2}\circ\varphi_1\times\varphi_2)(X_1,Y_2)=0$.

7. c) Both G and H are integrable iff

$$\begin{aligned} &N_{\varphi_\alpha}(X_\alpha,Y_\alpha)=0,\alpha=1,2,\ \ N_{\varphi_1\varphi_2}(X_1,Y_2)=0\\ &\text{or } N_{\varphi_\alpha}(X_\alpha,Y_\alpha)=0,\ \ N_{\varphi_\alpha}(X_1,Y_2)=0,\alpha=1,2. \end{aligned} \tag{5.8}$$

Definition 5.1. An almost CR-structure (D, J) on a manifold M is a CR-structure [1] if for any $X, Y \in D$ one has

$$\begin{aligned} &a)\ [JX, Y] + [X, JY] \in D, \\ &b)[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0. \end{aligned} \tag{5.9}$$

One remarks that a) is equivalent with

$$c)\ [JX, JY] - [X, Y] \in D.$$

From here and from 5.1 it results

Theorem 5.2. *1. The almost CR-structure (V_1, φ'_1) is a CR-structure iff*

$$N_{\varphi_1\varphi_2}(X_1, Y_1) = (N\varphi_1 + N\varphi_2)(X_1, Y_1) = 0. \tag{5.10}$$

2. The almost CR-structure (V_2, φ'_2) is a CR-structure iff

$$N_{\varphi_1\varphi_2}(X_2, Y_2) = (N\varphi_1 + N\varphi_2)(X_2, Y_2) = 0. \tag{5.11}$$

3. V_1 is involutive and (V_1, φ'_1) is a CR-structure iff

$$N\varphi_2(X_1, Y_1) = 0, \ \ N\varphi'_1(X_1, Y_1) = 0. \tag{5.12}$$

4. V_2 is involutive and (V_2, φ'_2) is a CR-structure iff

$$N\varphi_1(X_2, Y_2) = 0, \ \ N\varphi'_2(X_2, Y_2) = 0. \tag{5.13}$$

5. Both V_1 and V_2 are involutive and $(V_1, \varphi'_1), (V_2, \varphi'_2)$ are CR-structures iff

$$N_F = 0, \ \ N_{\varphi'_\alpha}(X_\alpha, Y_\alpha) = 0, \alpha = 1, 2. \tag{5.14}$$

Definition 5.2. An apbc-structure (F, G, H) is called integrable if there exists an atlas on M so that the associated natural bases are adapted bases for this structure.

Theorema 5.3. *An apbc-structure (F, G, H) is integrable iff one of the following conditions holds*

$$\begin{aligned} &N_F = N_G = N_H = 0; N_F = N_{\varphi'_1} = N_{\varphi'_2} = 0, \\ &N_{\varphi_1\varphi_2}(X_1, Y_2) = 0;\ N_{\varphi_1} = N_{\varphi_2} = 0. \end{aligned} \tag{5.15}$$

Proof. If the apbc-structure (F, G, H) is integrable, then there exists an atlas on M so that in the associated natural bases, the tensor fields $F, G, H, \varphi_1, \varphi_2, \varphi'_1, \varphi'_2$ are given by (2.8) and hence all the conditions 5.15 are satisfied.

Conversely, if $N_F = N_G = N_H = 0$, then from $N_F = 0$, it results (see [11]), that the distributions V_1 and V_2 are involutive and so there exists an atlas on M so that the leaves of V_1 are given locally by $x^a = const, a = 1, 2, \ldots, 2n_2$ and x^i, with $i = 1, 2, \ldots, 2n_1$, are the coordinates on them. The leaves of V_2 are given by $x^i = const$ and x^a are the local coordinates on them. Hence in the natural bases associated to this atlas, F is given by (2.8). Then from the integrability of G and H, [8], it follows $N_G(X_1, Y_1) = 0, N_H(X_2, Y_2) = 0$, which give us $N_{\varphi'_1} = N_{\varphi'_2} = 0$, i.e. the ac-structures φ'_1 and φ'_2 on the leaves of V_1 and V_2 respectively, are integrable. Therefore we can take a new atlas on M with the new coordinates of the form $s^p = s^p(x^i), t^p = t^p(x^i), p = 1, 2, \ldots, n_1, i = 1, 2, \ldots, 2n_1$ on the leaves of V_1 and $u^\alpha = u^\alpha(x^a), v^\alpha = v^\alpha(x^a), \alpha = 1, 2 \ldots, n_2, a = 1, 2, \ldots, 2n_2$ on the leaves of V_2, so that in these coordinates φ'_1 and φ'_2 and hence $F, G, H, \varphi_1, \varphi_2$ will be given by (2.8). As from the conditions 5.15_2 or 5.15_3 it follows $N_F = N_G = N_H = 0$, the theorem is proved.

Theorem 5.4. *The apbc-structure (F, G, H) is integrable iff there exists on M a symmetric FGH-connection.*

Proof. If the apbc-structure (F, G, H) is integrable, from the integrability of F it follows (see [11]), that exists a symmetric F-connection ∇^0 on M. Then, considering the connection

$$\nabla_X = \frac{1}{2}(\nabla^0_X - G \circ \nabla^0_X \circ G), \tag{5.16}$$

i.e. the conjugate of ∇^0 with respect to G, we obtain $\nabla F = \nabla G = \nabla H = 0$. Hence ∇ is a (F, G, H)-connection. For the torsion of ∇ we get

$$T(X, Y) = \frac{1}{2}[(\nabla^0_X G)(GY) - (\nabla^0_Y G)(GX)]. \tag{5.17}$$

But ∇ being a G-connection, from [8], we have for N_G

$$N_G(X, Y) = T(X, Y) + G(T(GX, Y)) + G(T(X, GY)) - T(GX, GY), \tag{5.18}$$

and substituting T from 5.17, we obtain finally,

$$N_G(X,Y) = 2T(X,Y). \tag{5.19}$$

Hence, G being integrable, one has $N_G = 0$ and so $T = 0$, i.e. ∇ is a symmetric F, G, H-connection.

Conversely, if there exists on M a symmetric (F, G, H)-connection ∇, then from the expressions 5.18, for N_G and the similar for N_F and N_H, it follows $N_F = N_G = N_H = 0$, i.e. the apbc-structure (F, G, H) is integrable. From the previous Theorem, it results.

Theorem 5.5. *For a Riemannian mapc-structure (F, G, H, g_1) on M, with the apbc-structure (F, G, H) integrable, one obtains;*

1. (F, g_1) and (F, g_2) are respectively Riemannian and pseudo- Riemannian locally product structures.

2. $(G, g_1), (H, g_1)$ and $(G, g_2), (H, g_2)$ are respectively Hermitian and indefinite Hermitian structures.

3. (φ'_1, γ'_1) and (φ'_2, γ'_2) are Hermitian structures on the leaves of the distributions V_1 and V_1 respectively.

6. Integrability for the almost symplectic structures ω_1 and ω_2. For the exterior differential of the as-structures ω_1 and ω_2, taking $X_\alpha, Y_\alpha, Z_\alpha \in V_\alpha, \alpha = 1, 2$, we obtain

$$\begin{aligned}
&d\omega_1(X_1, Y_1, Z_1) = d\psi_1(X_1, Y_1, Z_1),\\
&3d\omega_1(X_1, Y_1, Z_2) = (\mathcal{L}_{Z_2}\psi_1)(X_1, Y_1) - \psi_2(Z_2, [X_1, Y_1]),\\
&3d\omega_1(X_1, Y_2, Z_2) = (\mathcal{L}_{X_1}\psi_2)(Y_2, Z_2) + \psi_1(X_1, [Y_2, Z_2]),\\
&d\omega_1(X_2, Y_2, Z_2) = d\psi_2(X_2, Y_2, Z_2).\\
&d\omega_2(X_1, Y_1, Z_1) = d\psi_1(X_1, Y_1, Z_1),\\
&3d\omega_2(X_1, Y_1, Z_2) = (\mathcal{L}_{Z_2}\psi_1)(X_1, Y_1) + \psi_2(Z_2, [X_1, Y_1])\\
&3d\omega_2(X_1, Y_2, Z_2) = -(\mathcal{L}_{X_1}\psi_2)(Y_2, Z_2) + \psi_1(X_1, [Y_2, Z_2]),\\
&d\omega_2(X_2, Y_2, Z_2) = -d\psi_2(X_2, Y_2, Z_2).
\end{aligned} \tag{6.1}$$

From here it results

Theorem 6.1. *1. The as-structure ω_1 is integrable iff*

$$\begin{aligned}
&d\psi_1(X_1, Y_1, Z_1) = 0, (\mathcal{L}_{Z_2}\psi_1)(X_1, Y_1) = \psi_2(Z_2, [X_1, Y_1]),\\
&(\mathcal{L}_{X_1}\psi_2)(Y_2, Z_2) = -\psi_1(X_1, [Y_2, Z_2]), d\psi_2(X_2, Y_2, Z_2) = 0.
\end{aligned} \tag{6.2}$$

2. The as-structure ω_2 is integrable iff

$$
\begin{aligned}
&d\psi_1(X_1, Y_1, Z_1) = 0, \;\; \mathcal{L}_{Z_2}\psi_1(X_1, Y_1) = -\psi_2(Z_2, [X_1, Y_1]),\\
&\mathcal{L}_{X_1}\psi_2(Y_2, Z_2) = \psi_1(X_1, [Y_2, Z_2]), \;\; d\psi_2(X_2, Y_2, Z_2) = 0.
\end{aligned} \tag{6.3}
$$

3. Both ω_1 and ω_2 are integrable iff

$$
\begin{aligned}
&N_F = 0, \;\; d\psi_\alpha(X_\alpha, Y_\alpha, Z_\alpha) = 0, \alpha = 1, 2,\\
&\mathcal{L}_{Z_2}\psi_1(X_1, Y_1) = \mathcal{L}_{X_1}\psi_2(Y_2, Z_2) = 0.
\end{aligned} \tag{6.4}
$$

It results from here

Theorem 6.2. *If for a Riemannian mapbc-structure (F, G, H, g_1), the associated 2-forms ω_1 and ω_2 are integrable, then:*

1. (F, g_1) and (F, g_2) are respectively Riemannian and pseudo- Riemannian locally product structures.

2. $(G, g_1), (H, g_1)$ and $(G, g_2), (H, g_2)$ are respectively almost Kähler and indefinite almost Kähler structures.

3. (φ'_1, γ'_1) and (φ'_2, γ'_2) are almost Kähler structures on the leaves of V_1 and V_2 respectively.

4. Each vector field $Z_2 \in V_2$ (resp. $X_1 \in V_1$) generates a 1-parameter group of symplectomorphisms between the leaves of V_1 (resp. V_2).

In particular, from Theorem 5.2 and 6.2 and from [8,II,p.148] it follows

Theorem 6.3 *If for a Riemannian mapbc-structure (F, G, H, g_1), on M, the structures almost complex G, H and almost symplectic ω_1, ω_2 are integrable then:*

1. (F, g_1) and (F, g_2) are respectively Riemannian and pseudo- Riemannian locally decompozable structures.

2. $(G, g_1), (H, g_1)$ and $(G, g_2), (H, g_2)$ are respectively Kahler and indefinite Kähler structures.

3. (φ'_1, γ'_1) and (φ'_2, γ'_2) are Kähler structures on the leaves of V_1 and V_2 respectively.

7. Example. Let N be a manifold, $M = TN$ the total space of the tangent bundle $\pi : TN \to N$ and $VTN = Ker\ T\pi$ the vertical subbundle of TN. Denote by $(x^i), (x^i, y^i)$ the local coordinates on N and TN and by $(\partial i), (\partial i, \dot{\partial} i)$ the corresponding local bases, where $\partial_i = \frac{\partial}{\partial x^i}; \dot{\partial}_i = \frac{\partial}{\partial y^i}, i =$

$1, 2, \ldots, n$. Also we denote by $(d^i), (d^i, \dot{d}^i)$, where $d^i = dx^i, \dot{d}^i = dy^i$, the dual local bases on N and TN. Setting for each 1-form $\alpha \in \mathcal{D}_1(N)$, given locally by $\alpha(x) = \alpha_i(x)d^i, \gamma\alpha(z) = \alpha_i(x)y^i$, where $z = (x, y) \in T_xN$, we obtain a class of functions on TN with the property that every vector field $A \in \mathcal{D}^1(TN)$ is uniquely determined by its values on these functions. The mappings γ may be extended to tensor fields $S \in \mathcal{D}^1_1(N)$ by putting

$$\gamma S(\gamma\alpha) = \gamma(\alpha \circ S), \quad \forall \alpha \in \mathcal{D}_1(N). \tag{7.1}$$

Locally, if $S(x) = S^i_j(x)\partial_i \otimes d^j$, then $\gamma S(z) = S^i_j(x)y^j\partial_i$, hence γS is a vertical vector field on TN. Let then ∇ be a linear connection and X a vector field on N. Setting

$$X^h(\gamma\alpha) = \gamma(\nabla_X\alpha), X^v(\gamma\alpha) = \alpha(X) \circ \pi, \ \ \forall \alpha \in \mathcal{D}_1(N), \tag{7.2}$$

we obtain two vector fields on TN called respectively the horizontal and the vertical lifts of X. We have the following useful relations.

$$\begin{aligned} &f^h = f^v = f \circ \pi, (fX)^h = f^hX^h, (fX)^v = f^vX^v \\ &\left[X^h, Y^h\right] = [X, Y]^h - \gamma R_{XY}, \ \ \left[X^h, Y^v\right] = (\nabla_X Y)^v, [X^v, Y^v] = 0, \end{aligned} \tag{7.3}$$

where $f \in C^\infty(N), X, Y \in \mathcal{D}^1(N)$ and R is the curvature tensor of ∇.

Setting

$$F(X^h) = X^h, \ \ F(X^v) = -X^v, \ \ \forall X \in \mathcal{D}^1(N), \tag{7.4}$$

we obtain an ap-structure F on TN, whose $+1$ and -1 eigendistributions (subbundles) are respectively the horizontal distribution $V_1 = HTN$ associated to connection ∇ and the vertical distribution $V_2 = VTN$ of the tangent bundle TN. For $f \in \mathcal{D}^1_1(N)$ and $g \in \mathcal{D}^0_2(N)$, we define the horizontal (h) and vertical (v) lifts by

(7.5)
$$\begin{aligned} &f^h(X^h) = f(X)^h, f^h(X^v) = 0, ; \ \ f^v(X^h) = 0, \ \ f^v(X^v) = f(X)^v; \\ &g^h(X^h, Y^h) = g(X, Y)^v, g^h(X^h, Y^v) = g^h(X^v, Y^h) = g^h(X^v, Y^v) = 0, \\ &g^v(X^h, Y^h) = g^v(X^h, Y^v) = g^v(X^v, Y^h) = 0, g^v(X^v, Y^v) = g(X, Y)^v. \end{aligned}$$

Let now (f, g) be an almost Hermitian structure on N and $\omega = g \circ I \times f$ the associated 2-form. Setting

$$F = I^h - I^v, \ \ G = f^h + f^v, \ \ H = f^h - f^v, \tag{7.6}$$

we obtain the ap-structure F given by (7.4) and two ac-structures G and H, which satisfy the conditions (2.1), i.e. determine an apbc-structure on TN. Putting then

$$g_1 = g^h + g^v, g_2 = g^h - g^v, \omega_1 = \omega^h + \omega^v, \omega_2 = \omega^h - \omega^v, \tag{7.7}$$

we get that g_1, g_2 determine Riemannian and pseudo-Riemannian structures respectively and ω_1, ω_2 as-structures on TN, which satisfy the conditions (3.1) and (3.2). Hence, we have

Theorem 7.1 *Given an almost Hermitian structure (f, g) with the associated 2-form ω and a linear connection ∇ on N, one obtains by the formulas (7.6) and (7.7) a Riemannian mapbc-structure (F, G, H, g_1), with the associated metric g_2 and two as-structures (ω_1, ω_2) on the manifold TN. The pair of the associated supplementary cc-structures is given by $\varphi_1 = f^h, \varphi_2 = f^v$, the pairs of induced almost Hermitian structures on the distributions $V_1 = HTM$ and $V_2 = VTM$ by $(f^h, g^h)/V_1, (f^v, g^v)/V_2$ and the supplementary almost CR-structures by $(V_1, f^h/V_1), (V_2, f^v/V_2)$.*

For a connection ∇ on N we define the diagonal lift D, (see [6]), by

$$\begin{aligned} &D_{X^h}Y^h = (\nabla_X Y)^h, D_{X^h}Y^v = (\nabla_X Y)^v, \\ &D_{X^v}Y^h = D_{X^v}Y^v = 0, \forall X, Y \in \mathcal{D}^1(N). \end{aligned} \tag{7.8}$$

The nonvanishing components of the torsion and the curvature tensor fields of D, are given by

$$\begin{aligned} &\mathcal{T}(X^h, Y^h) = T(X, Y)^h + \gamma R_{XY}, \\ &\mathcal{R}_{X^hY^h}Z^h = (R_{XY}Z)^h, R_{X^hY^h}Z^v = (R_{XY}Z)^v, \end{aligned} \tag{7.9}$$

where T and R are the torsion and curvature tensors of ∇. After that, for the covariant derivatives with respect to D, of F, G, H, g_α and $\omega_\alpha, \alpha = 1, 2$ we obtain

$$\begin{aligned} &DF = 0; D_{X^h}G = (\nabla_X f)^h + (\nabla_X f)^v, D_{X^v}G = 0; \\ &D_{X^h}H = (\nabla_X f)^h - (\nabla_X f)^v, D_{X^v}H = 0; \\ &D_{X^h}g_1 = (\nabla_X g)^h + (\nabla_X g)^v, D_{X^v}g_1 = 0; \\ &D_{X^h}g_2 = (\nabla_X g)^h - (\nabla_X g)^v, D_{X^v}g_2 = 0; \\ &D_{X^h}\omega_1 = (\nabla_X \omega)^h + (\nabla_X \omega)^v, D_{X^v}\omega_1 = 0; \\ &D_{X^h}\omega_2 = (\nabla_X \omega)^h - (\nabla_X \omega)^v, D_{X^v}\omega_2 = 0. \end{aligned} \tag{7.10}$$

Hence, $DF = 0$ always, $DG = DH = 0$, iff $\nabla f = 0, Dg_\alpha = 0, \alpha = 1,2$ iff $\nabla g = 0$ and $D\omega_\alpha = 0, \alpha = 1,2$ iff $\nabla\omega = 0$. So we have

Theorem 7.2. *The diagonal lift D on TN, for a connection ∇ on N, is a (F, G, H, g_1)-connection iff ∇ is a (f, g)-connection, i.e. iff ∇ is given by*

$$\nabla = \psi_f \circ \psi_g(\nabla^0) + \chi_f \circ \chi_g(\tau), \tag{7.11}$$

with $\nabla^0 \in C(N)$ fixed and $\tau \in \mathcal{D}_2^1(N)$ arbitrary.

For the Nijenhuis tensors of F, G and H one obtains

(7.12)

$$N_F(X^h, Y^h) = 4\gamma R_{XY}, N_F(X^h, Y^v) = 0, N_F(X^v, Y^v) = 0;$$
$$N_G(X^h, Y^h) = N_f(X, Y)^h + \gamma[R_{XY} - R_{fXfY} + f \circ (R_{fXY} + R_{XfY})],$$
$$N_G(X^h, Y^v) = [(\nabla_{fX} f - f \circ \nabla_X f)(Y)]^v, N_G(X^v, Y^v) = 0;$$
$$N_H(X^h, Y^h) = N_f(X, Y)^h + \gamma[R_{XY} - R_{fXfY} - f \circ (R_{fXY} + R_{XfY})],$$
$$N_H(X^h, Y^v) = -[(\nabla_{fX} f + f \circ \nabla_X f)(Y)]^v, N_H(X^v, Y^v) = 0.$$

From here it results.

Theorem 7.3. 1. The ap-structure F is integrable iff $R = 0$;

2. The ac-structure G is integrable iff

$$\begin{aligned} &N_f = 0, \nabla_{fX} f - f \circ \nabla_X f = 0,\\ &R_{XY} - R_{fXfY} + f \circ (R_{fXY} + R_{XfY}) = 0; \end{aligned} \tag{7.13}$$

3. The ac-structure H is integrable iff

$$\begin{aligned} &N_f = 0, \nabla_{fX} f + f \circ \nabla_X f = 0,\\ &R_{XY} - R_{fXfY} - f \circ (R_{fXY} + R_{XfY}) = 0; \end{aligned} \tag{7.14}$$

4. Both the ac-structure G and H are integrable iff

$$N_f = 0, \nabla f = 0, R_{XY} - R_{fXfY} = 0; \tag{7.15}$$

5. The apbc-structure (F, G, H) is integrable iff

$$N_f = 0, \nabla f = 0, R = 0. \tag{7.16}$$

For the exterior derivative of the 2-forms ω_1 and ω_2 we obtain

$$d\omega_1(X^h, Y^h, Z^h) = d\omega(X, Y, Z)^h, 3d\omega_1(X^h, Y^h, Z^v) = -\gamma(i_Z\omega \circ R_{XY}),$$
$$3d\omega_1(X^h, Y^v, Z^v) = (\nabla_X\omega(Y, Z))^v, d\omega_1(X^v, Y^v, Z^v) = 0;$$
$$d\omega_2(X^h, Y^h, Z^h) = d\omega(X, Y, Z)^h, 3d\omega_2(X^h, Y^h, Z^v) = \gamma(i_Z\omega \circ R_{XY}),$$
$$3d\omega_2(X^h, Y^v, Z^v) = -(\nabla_X\omega)(Y, Z)^v, d\omega_2(X^v, Y^v, Z^v) = 0.$$

So, one has

Theorem 7.4. *The 2-forms $\omega_\alpha, \alpha = 1, 2$ are simultaneous integrable, namely iff*

$$d\omega = 0, \nabla\omega = 0, R = 0. \tag{7.18}$$

From (7.16) and (7.18) one obtains.

Theorem 7.5. *The apbc-structure (F, G, H) and the as-structures ω_1, ω_2 are simultaneous integrable iff*

$$N_f = d\omega = 0, \nabla f = \nabla\omega = 0, R = 0, \tag{7.19}$$

with other words iff (f, g) is a Kahler structure and ∇ $a(f, g)$-connection with vanishing curvature on N.

In particular, these conditions are satisfied if (f, g) is a Kahler structure with vanishing curvature on N and ∇ the Levi-Civita connection of g.

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Faculty of Mathematics,
"Al. I. Cuza" University,
Bd. Carol I, No. 11,
700506, Iasi,
ROMÂNIA,