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# ON ALMOST BIPRODUCT COMPLEX MANIFOLDS*

BY

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**Abstract.** One defines the almost biproduct complex (abpc) structure and one analyzes its equivalence with other structures on a manifold. One studies then the metrics and connections compatible with such a structure, the involutivity of the associated distributions and the integrability of these structures. An example of a metric (abpc)-structure on the tangent bundle of a Riemannian manifold is also given.

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**1. Introduction.** Let $F$ and $P$ be two $(1,1)$-tensor fields on a manifold $M$ so that the endomorphisms defined by these are: both almost product, or one almost product and other almost complex, or both almost complex, which commute or anticommute. With the triplet $(F, P, J = P \circ F)$ we can form the following four structures:

1) $F^2 = P^2 = J^2 = F \circ P \circ J = I$,

2) $F^2 = P^2 = -J^2 = F \circ P \circ J = I$,

3) $-F^2 = P^2 = J^2 = F \circ P \circ J = -I$,

4) $F^2 = P^2 = J^2 = F \circ P \circ J = -I$,

called respectively: almost hyperproduct (ahp), almost biproduct complex (abpc), almost product bicomplex (apbc) and almost hypercomplex (ahc).

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Along the time all these structures, with different others denominations, were considered, together or separately, by: LIBERMAN [10], CRUCEANU [3,6,7], BONOME, CASTRO, GARCIA-RIO, HERVELLA and MATSUSHITA [2], HSU [8], MACSYM and ZMUREK [11], SALAMON [13], SANTAMARIA [14,15] VIDAL and VIDAL COSTA [16], YANO and AKO [17] and many others.

Continuing the recent studies for two of these structures [6,7], in this paper, we consider the (abpc)-structures. Firstly, we give a new definition for an (abpc)-structure and we analyze its equivalence with many other important structures on a manifold. We study then, metrics, symplectic structures and linear connections compatible with such a structure, the involutivity of the associated distributions and the integrability of the (abpc)-structures, using some canonical compatible connections. An example of a metric (abpc)-structure on the total space of the tangent bundle of a Riemannian manifold is also given.

**2. Almost biproduct complex structures and equivalent structures.** Let $M$ be a paracompact and connected manifold, $\mathcal{F}(M)$ the ring of real functions, $\mathcal{T}_q^p(M)$ the $\mathcal{F}(M)$-module of $(p,q)$-tensor fields and $\mathcal{T}(M)$ the $\mathcal{F}(M)$-tensor algebra of $M$, all in the category of $C^\infty$-manifolds. For a distribution $W$ on $M$ we denote by $\mathcal{T}^1(M,W)$, the $\mathcal{F}(M)$-module of $C^\infty$-sections in the subbundle $W$.

**Definition 2.1.** *An almost biproduct complex (abpc)-structure on the manifold $M$ is a triplet $(F,P,J)$ of $(1,1)$-tensor fields which satisfy*

$$F^2 = P^2 = -J^2 = F \circ P \circ J = I. \tag{2.1}$$

*An almost biproduct complex manifold is a manifold endowed with an (abpc)-structure.*

It is easy to see that the conditions (2.1) are equivalent with the property that $F$ and $P$ are almost product (ap)-structures and $J$ is an almost complex (ac)-structure on $M$, which satisfy the relations

$$F \circ P = -P \circ F = -J, \quad P \circ J = -J \circ P = F, \quad J \circ F = -F \circ J = P. \tag{2.2}$$

A structure $(F,P,J)$ which satisfies the conditions (2.1) was called by different authors; almost quaternionic of the second kind, or almost antiquaternionic, or almost paraquaternionic structure.

Considering the projectors $F^{\pm}$ of $F$ and $P^{\pm}$ of $P$, given by

$$F^{+} = \frac{I+F}{2}, \;\; F^{-} = \frac{I-F}{2}, \;\; P^{+} = \frac{I+P}{2}, \;\; P^{-} = \frac{I-P}{2}, \tag{2.3}$$

and the eigendistributions (subbundles) of $TM$

$$V_1 = F^{+}, \;\; V_2 = F^{-}; \;\; V_3 = P^{+}, \;\; V_4 = P^{-}, \tag{2.4}$$

one obtains $V_2 = P(V_1)$ and $V_4 = F(V_3)$. Hence, $dim V_i = n$, $i = 1, 2, 3, 4$, $dim\, M = 2n$ and $Tr\, F = Tr\, P = 0$. That is, $F$ and $P$ are almost paracomplex (apc)-structures on $M$, which anticommute.

**Definition 2.2.** *A pair $(F, P)$ of anticommuting almost product structures on a manifold $M$ is called an almost biparacomplex structure.*

From the considerations in the above it follows:

**Proposition 2.1.** *If $(F, P, J)$ is an $(abpc)$-structure on $M$, then $(F, P)$ is an almost biparacomplex structure. Conversely if $(F, P)$ is an almost biparacomplex structure on $M$, then $(F, P, J = F \circ P)$ is an $(abpc)$-structure.*

**Definition 2.3.** *A pair $(F, J)$ formed by an $(ap)$-structure $F$ and an $(ac)$-structure $J$, which anticommute, is called an almost product complex $(apc)$-structure on $M$.*

One has

**Proposition 2.2.** *If $(F, P, J)$ is an $(abpc)$-structure on $M$, then the pairs $(F, J)$ and $(P, J)$ are $(apc)$-structures. Conversely if $(F, J)$ is an $(apc)$-structure on $M$, then $(F, P = J \circ F, J)$ is an $(abpc)$-structure.*

**Definition 2.4.** *An almost tangent $(at)$-structure on $M$ is an endomorphism $A$ of $TM$, with the properties $A^2 = 0, Ker\, A = Im\, A$. An almost bitangent $(abt)$-structure on $M$ is a pair $(A, B)$ of $(at)$-structures so that $A \circ B + B \circ A = I$.*

From (2.1) one obtains:

**Proposition 2.3.** *If $(F, P, J)$ is an $(abpc)$-structure on $M$ then setting*

$$A = \frac{1}{2}(F + J), \;\; B = \frac{1}{2}(F - J); \;\; C = \frac{1}{2}(P + J), \;\; D = \frac{1}{2}(P - J), \tag{2.5}$$

*the pairs* $(A, B)$ *and* $(C, D)$ *are* $(abt)$*-structures on* $M$. Conversely, if $(A, B)$ is an (abt)-structure on $M$, then setting

$$F = A + B, \quad J = A - B, \quad P = J \circ F, \tag{2.6}$$

the triplet $(F, P, J)$ is an (abpc)-structure.

**Definition 2.5.** *An* $\alpha$*-structure on the manifold* $M$ *is a triplet* $(V_1, V_2, V_3)$ *of distributions on* $M$*, by twos supplementary.*

Considering the eigendistributions $V_i, i = 1, 2, 3, 4$ associated to an (abpc)-structure, one obtains:

**Proposition 2.4.** *If* $(F, P, J)$ *is an* $(abpc)$*-structure on* $M$ *and* $V_1 = F^+, V_2 = F^-, V_3 = P^+, V_4 = P^-$ *are the eigendistributions of* $F$ *and* $P$*, then the triplets* $(V_1, V_2, V_3), (V_2, V_3, V_4), (V_3, V_4, V_1), (V_4, V_1, V_2)$ *are* $\alpha$*-structures. Conversely, if* $(V_1, V_2, V_3)$ *is an* $\alpha$*-structure on* $M$*, then putting* $F^+ = V_1, F^- = V_2, P^+ = V_3, P^- = F(V_4)$ *the triplet* $(F, P, J = P \circ F)$ *is an* $(abpc)$*-structure.*

**Definition 2.6.** *A* $\beta$*-structure on* $M$ *is a pair* $(H, W)$ *where* $H$ *is an* $(ap)$*-structure and* $W$ *a distribution on* $M$*, so that* $TM = W \oplus H(W)$*.*

From (2.4) it results:

**Proposition 2.5.** *If* $(F, P, J)$ *is an* $(abpc)$*-structure on* $M$*, then the pairs* $(F, P^+), (F, P^-), (P, F^+), (P, F^-)$ *are* $\beta$*-structures. Conversely, if* $(F, W)$ *is a* $\beta$*-structure on* $M$*, then setting* $P^+ = W$ *and* $P^- = F(W)$*, the triplet* $(F, P, J = P \circ F)$ *is an (abpc)-structure and one has*

$$P(X + FY) = X - FY, \; J(X + FY) = Y - FX, \quad X, Y \in \mathcal{T}^1(M, W). \tag{2.7}$$

**Definition 2.7.** *A* $\gamma$*-structure on* $M$ *is a pair* $(H, W)$*, where* $H$ *is an* $(ac)$*-structure and* $W$ *a distribution on* $M$ *so that* $TM = W \oplus JW$*.*

From (3.4) one obtains:

**Proposition 2.6.** *If* $(F, P, J)$ *is an* $(abpc)$*-structure on* $M$*, then the pairs* $(J, F^+), (J, F^-), (J, P^+), (J, P^-)$ *are* $\gamma$*-structures. Conversely, if* $(J, W)$ *is a* $\gamma$*-structure on* $M$*, then setting*

$$F(X + JY) = X - JY, \quad P(X + JY) = Y + JX, \quad X, Y \in \mathcal{T}^1(M, W), \tag{2.8}$$

*the triplet $(F, P, J)$ is an $(abpc)$-structure.*

**Definition 2.8.** *A $\delta$-structure on $M$ is a pair $(H, W)$, where $H$ is an $(at)$-structure and $W$ a distribution on $M$ so that $TM = W \oplus HW$.*

**Proposition 2.7.** *If $(F, P, J)$ is an $(abpc)$-structure on $H$, then the pairs $(\frac{F+J}{2}, P^+), (\frac{F-J}{2}, P^-), (\frac{P+J}{2}, F^+), (\frac{P-J}{2}, F^-)$ are $\delta$-structures. Conversely, if $(A, W)$ is a $\delta$-structure on $M$, then setting*

$$
\begin{aligned}
&F(X + AY) = X - AY,\ P(X + AY) = Y + AX, \\
&J(X + AY) = -Y + AX,\ X, Y \in \mathcal{T}^1(M, W),
\end{aligned} \tag{2.9}
$$

*the triplet $(F, P, J)$ is an (abpc)-structure.*

Summarizing the previous considerations one obtains:

**Theorem 2.1.** *An $(apbc)$-structure on a manifold is equivalent with each of the following structures: almost biparacomplex, almost product complex, almost bitangent, $\alpha, \beta, \gamma$ and $\delta$.*

**Definition 2.9.** *An adapted local basis for the apbc-structure $(F, P, J)$ on $M$ is a local basis $(e_i, Pe_i), i = 1, 2, \ldots, n$, where $(e_i)$ is a local basis on $V_1 = F^+$.*

In such a basis the tensor fields $F, P, J$ have the matrices

$$
F = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix},\quad P = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix},\quad J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \tag{2.10}
$$

From here it follows:

**Theorem 2.2.** *The structural group for the tangent bundle of a 2n-dimensional manifold $M$, endowed with an $(abpc)$-structure, is reducible, to the diagonal subgroup of the direct product $GL(n, R) \times GL(n, R)$.*

**3. Metric and symplectic structures compatible with an (abpc)-structure.** Let $h$ be a Riemannian metric on the (abpc)-manifold $M$ and

$$
\begin{aligned}
&g_1 = h \circ (I \times I + F \times F + P \times P + J \times J), \\
&g_2 = g_1 \circ I \times F,\ g_3 = g_1 \circ I \times P,\ \omega = g_1 \circ I \times J.
\end{aligned} \tag{3.1}
$$

From here and (2.1) it results the following table of compatibilities:

(3.2)

| $\circ$ | F | P | J |
|---|---|---|---|
| $g_1$ | $g_1 \circ F \times F = g_1$<br>$g_1 \circ I \times F = g_2$ | $g_1 \circ P \times P = g_1$<br>$g_1 \circ I \times P = g_3$ | $g_1 \circ J \times J = g_1$<br>$g_1 \circ I \times J = \omega$ |
| $g_2$ | $g_2 \circ F \times F = g_2$<br>$g_2 \circ I \times F = g_1$ | $g_2 \circ P \times P = -g_2$<br>$g_2 \circ I \times P = -\omega$ | $g_2 \circ J \times J = -g_2$<br>$g_2 \circ I \times J = -g_3$ |
| $g_3$ | $g_3 \circ F \times F = -g_3$<br>$g_3 \circ I \times F = \omega$ | $g_3 \times P \times P = g_3$<br>$g_3 \circ I \times P = g_1$ | $g_3 \circ J \times J = -g_3$<br>$g_3 \circ I \times J = g_2$ |
| $\omega$ | $\omega \circ F \times F = -\omega$<br>$\omega \circ I \times F = g_3$ | $\omega \circ P \times P = -\omega$<br>$\omega \circ I \times P = -g_2$ | $\omega \circ J \times J = \omega$<br>$\omega \circ I \times J = -g_1$ |

Taking into account (2.1), (3.1) and (3.2), one obtains that $g_1$ is a Riemannian metric, $g_2, g_3$ are neutral metrics and $\omega$ is an almost symplectic (as)-structure on $M$.

**Definition 3.1.** *We call the quatriplet $(F, P, J, g_1)$, which satisfies the conditions* (2.1), (3.1) *and* (3.2), *a Riemannian metric almost biproduct complex* (*mabpc*)*-structure on $M$ and $g_2, g_3, \omega$ the associated neutral metrics and almost symplectic structure.*

From the previous considerations we can state the following result:

**Theorem 3.1.** *A Riemannian* (*mabpc*)*-structure $(F, P, J, g_1)$ determines on $M$:*

a) *two Riemannian almost paracomplex structures $(F, g_1)$ and $(P, g_1)$, with the associated neutral metrics $g_2$ and $g_3$ respectively,*

b) *two neutral metric almost paracomplex structures $(F, g_2)$ and $(P, g_3)$ with the associated Riemannian metric $g_1$,*

c) *two almost para-Hermitian structures $(F, g_3)$ and $(P, g_2)$, with the associated almost symplectic 2-forms $\omega$ and $-\omega$ respectively,*

d *an almost Hermitian structure $(J, g_1)$ with the associated almost symplectic 2-form $\omega$ and*

e) *two almost anti-Hermitian structures $(J, g_2)$ and $(J, g_3)$ with the associated neutral metrics $-g_3$ and $g_2$ respectively.*

Setting $g = g_1/V_1 \times V_1$, one obtain from (3.1) and (3.2), for $X_\alpha, Y_\alpha \in \mathcal{T}^1(M, V_\alpha), \alpha = 1, 2,,$

(3.3)

$$g_1(X_1, Y_1) = g(X_1, Y_1),\ \ g_1(X_1, Y_2) = 0,\ \ g_1(X_2, Y_2) = g(PX_2, PY_2),$$
$$g_2(X_1, Y_1) = g(X_1, Y_1),\ g_2(X_1, Y_2) = 0,\ g_2(X_2, Y_2) = -g(PX_2, PY_2),$$
$$g_3(X_1, Y_1) = 0,\ g_3(X_1, Y_2) = g(X_1, PY_2),\ g_3(X_2, Y_2) = 0,$$
$$\omega(X_1, Y_1) = 0,\ \omega(X_1, Y_2) = -g(X_1, PY_2),\ \omega(X_2, Y_2) = 0$$

and from here and Proposition 2.1 it results:

**Proposition 3.1.** *A Riemannian (mabpc)-structure $(F, P, J, g_1)$ on $M$ is uniquely determined by an almost biparacomplex structure $(F, P)$ and a Riemannian metric $g$ on the distribution $V_1 = F^+$.*

**Definition 3.2.** *An adapted local basis to the Riemannian (mabpc)-structure $(F, P, J, g_1)$ is an adapted basis $(e_i, Pe_i)$ to the (abpc)-structure $(F, P, J)$, where $(e_i)$ is an orthonormal basis on $V_1 = F^+$.*

In such a basis, the matrices, associated to structures $g_\alpha, \alpha = 1, 2, 3$ and $\omega$, coincide with the matrices of $I, F, P, J$ respectively. So one obtains

**Theorem 3.2.** *The structural group, for the tangent bundle of a 2n-dimensional manifold $M$, endowed with a Riemannian (mapbc)-structure, is reducible to the diagonal subgroup of the direct product $SO(n) \times SO(n)$.*

## 4. Connections compatible with an (abpc)-structure

**Definition 4.1.** *A linear connection $\nabla$ on $M$ is called compatible with the (abpc)-structure $(F, P, J)$ or is a $(F, P, J)$-connection iff*

$$\nabla F = \nabla P = \nabla J = 0. \tag{4.1}$$

Let $C(M)$ be the $\mathcal{F}(M)$-affine module of connections on $M$. Setting for $\nabla \in C(M)$, $\tau \in \mathcal{T}_2^1(M)$ and $X \in \mathcal{T}^1(M)$,

(4.2)
$$\psi_F(\nabla)_X = \frac{1}{2}(\nabla_X + F \circ \nabla_X \circ F),\ \chi_F(\tau)_X = \frac{1}{2}(\tau_X + F_0 \tau_X \circ F),$$
$$\psi_P(\nabla)_X = \frac{1}{2}(\nabla_X + P \circ \nabla_X \circ P),\ \chi_P(\tau)_X = \frac{1}{2}(\tau_X + P \circ \tau_X \circ P),$$
$$\psi_J(\nabla)_X = \frac{1}{2}(\nabla_X - J \circ \nabla_X \circ J),\ \chi_J(\tau)_X = \frac{1}{2}(\tau_X - J \circ \tau_X \circ J),$$

we obtain as in [6,7], the following:

**Proposition 4.1.** *The set* $C_{FPJ}(M)$ *of the connections on* $M$*, compatible with an* (*abpc*)*-structure* $(F,P,J)$*, is given by*

$$\nabla = \psi_F \circ \psi_P(\nabla^\circ) + \chi_F \circ \chi_P(\tau), \tag{4.3}$$

*where* $\nabla^0 \in C(M)$ *is fixed and* $\tau \in \mathcal{T}_2^1(M)$ *is arbitrary.*

Taking here $\tau = 0$, it follows that an (abpc)-structure $(F,P,J)$ associates to each connection $\nabla^0 \in C(M)$, a $(F,P,J)$-connection $\nabla = \psi_F \circ \psi_P(\nabla^\circ)$. This connection may be written in the form

$$\nabla_X = \frac{1}{4}\big(\nabla^0_X + F \circ \nabla^\circ_X \circ F + P \circ \nabla^\circ_X \circ F - J \circ \nabla^0_X \circ J\big),\ X \in \mathcal{T}^1(M), \tag{4.4}$$

i.e. $\nabla$ is the *mean connection* of $\nabla^0$ and its *conjugate connections* [6] with respect to $F, P$ and $J$.

**Definition 4.2.** *A connection* $\nabla$ *is compatible with the structure* $\alpha = (V_1, V_2, V_3)$ *iff it preserves by parallelism the distributions* $V_i, i = 1,2,3.$

**Definition 4.3.** *A connection* $\nabla$ *is compatible with one of the structures* $\beta, \gamma, \delta = (H, W)$ *iff it is a* $H$*-connection which preserves the distribution* $W$*.*

It is not difficult to prove

**Proposition 4.2.** *A connection* $\nabla$ *on* $M$ *is compatible with the* (*abpc*)*-structure* $(F,P,J)$ *iff it satisfies one of the conditions:*

1. *The tensor fields from one of the pairs* $(F,P), (P,J), (J,F), (A,B)$ *are covariant constant,*
2. $\nabla$ *is compatible with one of the structure* $\alpha, \beta, \gamma, \delta$*.*

Setting now, for a connection $\nabla$ on $M$,

$$\overset{i}{\nabla}_X Y = F_i(\nabla_X Y),\ i = 1,2,\ \overset{i}{\nabla}_X Y = P_i(\nabla_X Y),\ i = 3,4,\ X, Y \in \mathcal{T}^1(M), \tag{4.5}$$

one finds that the operators $\overset{i}{\nabla}, i = 1,2,3,4$ are $\mathcal{F}(M)$-linear in the first argument, $R$-linear in the second and satisfy

$$\begin{aligned} \overset{i}{\nabla}_X(fY) &= X(f)F_i(Y) + f\overset{i}{\nabla}_X Y,\ i = 1,2, \\ \overset{i}{\nabla}_X(fY) &= X(f)P_i(Y) + f\overset{i}{\nabla}_X Y,\ i = 3,4, \end{aligned} \tag{4.6}$$

for $f \in \mathcal{F}(M)$ and $X, Y \in \mathcal{T}^1(M)$. It results from here that $(\overset{i}{\nabla}, F_i)$, $i = 1, 2$ and $(\overset{i}{\nabla}, P_i), i = 3, 4$ are *quasi-connections* in the sense of Otsuki [12]. The restrictions of $\overset{i}{\nabla}$, with respect to the second argument, to $\mathcal{T}^1(M, V_i)$ give the connections induced by $\nabla$ on the subbundles $V_i$ [6]. When $\nabla$ is compatible with the (abpc)-structure $(F, P, J)$, the connections $\overset{i}{\nabla}$ coincide with the restrictions of $\nabla$, with respect to second argument, to $\mathcal{T}^1(M, V_i), i = 1, 2, 3, 4$.

If $\nabla$ is an arbitrary connection on $M$ we can consider for the vector 1-forms $F_i, i = 1, 2$ and $P_i, i = 3, 4$, the exterior covariant derivatives with respect to $\nabla$ given by

$$\begin{aligned} dF_i(X, Y) &= \nabla_X(F_iY) - \nabla_Y(F_iX) - F_i[X, Y],\ i = 1, 2 \\ dP_i(X, Y) &= \nabla_X(P_iY) - \nabla_Y(P_iX) - P_i[X, Y],\ i = 3, 4. \end{aligned} \tag{4.7}$$

It is naturally to call the *torsion* of the connection $\overset{i}{\nabla}$, induced by $\nabla$ to $V_i$, the restriction $\overset{i}{T}$, of $dF_i$ and $dP_i$ to corresponding $\mathcal{T}^1(M, V_i) \times \mathcal{T}^1(M, V_i), i = 1, 2, 3, 4$. So, we have

$$\overset{i}{T}(X_i, Y_i) = \begin{cases} \nabla_{X_i}Y_i - \nabla_{Y_i}X_i - F_i[X_i, Y_i],\ i = 1, 2 \\ \nabla_{X_i}Y_i - \nabla_{Y_i}X_i - P_i[X_i, Y_i],\ i = 3, 4. \end{cases} \tag{4.8}$$

If $\nabla$ is compatible with the (abpc)-structure $(F, P, J)$, we get for $\overset{i}{T}$, as tensor fields on $M$, the expressions

$$\overset{i}{T} = \begin{cases} F_i \circ T \circ F_i \times F_i, & i = 1, 2 \\ P_i \circ T \circ P_i \times P_i, & i = 3, 4, \end{cases} \tag{4.9}$$

where $T$ is the torsion of $\nabla$.

In this case for the curvature of $\nabla$ one obtains

$$R_{XY} \circ F = F \circ R_{XY},\ \ R_{XY} \circ P = P \circ R_{XY}, \tag{4.10}$$

i.e. $R_{XY}$, as endomorphism of $TM$, preserves the distributions $V_i$, $i = 1, 2, 3, 4$.

From here it follows, for the curvatures $\overset{i}{R}$ of $\overset{i}{\nabla}$, considered as tensor fields on $M$,

$$\overset{i}{R}_{XY} = \begin{cases} R_{XY} \circ F_i, & i = 1,2 \\ R_{XY} \circ P_i, & i = 3,4 \end{cases} \tag{4.11}$$

i.e. the curvature of the induced connection $\overset{i}{\nabla}$ coincides with the restriction of the curvature $R$ of $\nabla$ to subbundle $V_i, i = 1,2,3,4$.

We have also the following result:

**Proposition 4.3.** *A connection $\nabla$ on $M$, compatible with the (abpc)-structure $(F, P, J)$, is uniquely determined by its restriction $\overset{1}{\nabla}$ (respectively $\overset{2}{\nabla}$) with respect to second argument, to $\mathcal{T}^1(M, V_1)$ (respectively $\mathcal{T}^1(M, V_2)$) or by one of the pairs of partial connections $(\overset{1}{\nabla}_{X_1}, \overset{2}{\nabla}_{X_2}), (\overset{1}{\nabla}_{X_2}, \overset{2}{\nabla}_{X_1})$, with $X_i \in \mathcal{T}^1(M, V_i)$, $i = 1,2$.*

Indeed setting, for $Y \in \mathcal{T}^1(M), Y = Y_1 + Y_2$ with $Y_i \in \mathcal{T}^1(M, V_i)$, $i = 1,2$, from $\nabla F = 0$ it follows $\nabla_X Y = \overset{1}{\nabla}_X Y_1 + \overset{2}{\nabla}_X Y_2$. Then, from $\nabla P = 0$ one obtains $\overset{2}{\nabla}_X = P \circ \overset{1}{\nabla}_X \circ P$ and so

$$\nabla_X Y = \overset{1}{\nabla}_X Y_1 + (P \circ \overset{1}{\nabla}_X \circ P)(Y_2) = (P \circ \overset{2}{\nabla}_X \circ P)(Y_1) + \overset{2}{\nabla}_X Y_2. \tag{4.12}$$

Putting then $X = X_1 + X_2$, one has $\overset{i}{\nabla}_X = \overset{i}{\nabla}_{X_1} + \overset{i}{\nabla}_{X_2}$, $i = 1,2$ and from $\overset{1}{\nabla}_X = P \circ \overset{2}{\nabla}_X \circ P$ it follows

$$\overset{1}{\nabla}_X = \overset{1}{\nabla}_{X_1} + P \circ \overset{2}{\nabla}_{X_2} \circ P = P \circ \overset{2}{\nabla}_{X_1} \circ P + \overset{1}{\nabla}_{X_2}. \tag{4.13}$$

From here, we obtain the following important result:

**Theorem 4.1.** *On a manifold $M$ endowed with an (abpc)-structure $(F, P, J)$ there exists a unique connection $\nabla$, with torsion $T$, satisfying the conditions*

$$\nabla F = \nabla P = 0, \ \ T \circ F_1 \times F_2 = 0, \tag{4.14}$$

*where $F_1$ and $F_2$ are the projectors of $F$.*

*Uniqueness.* Let $\nabla$ be a connection which satisfies (4.14). It results

$$T(X_1, X_2) = \overset{2}{\nabla}_{X_1} X_2 - \overset{1}{\nabla}_{X_2} X_1 - F_1[X_1, X_2] - F_2[X_1, X_2] = 0.$$

As $V_1$ and $V_2$ are supplementary, it follows from here,

$$\overset{1}{\nabla}_{X_2} X_1 = F_1[X_2, X_1],\ \overset{2}{\nabla}_{X_1} X_2 = F_2[X_1, X_2],\ \ \forall X_i \in \mathcal{T}^1(M, V_i), i = 1, 2.$$

But, from Proposition 4.3, $\nabla$ being determined by $\overset{1}{\nabla}_{X_2}$ and $\overset{2}{\nabla}_{X_1}$, it is unique.

*Existence.* Setting for $X_i, Y_i \in \mathcal{T}^i(M, V_i)$, $i = 1, 2$

$$\begin{aligned} &\nabla_{X_1} Y_1 = F_1 \circ P[X_1, PY_1],\ \nabla_{X_1} X_2 = F_2[X_1, X_2], \\ &\nabla_{X_2} X_1 = F_1[X_2, X_1],\ \nabla_{X_2} Y_2 = F_2 \circ P[X_2, PY_2], \end{aligned} \tag{4.5}$$

and using the relations $F_1 \circ P = P \circ F_2, F_2 \circ P = P \circ F_1$, one obtains that $\nabla$ is a connection on $M$ which satisfies the conditions (4.14).

Changing the order of $F$ and $P$ (and so of $(V_1, V_2)$ with $(V_3, V_4)$), we obtain another unique connection $\nabla'$ on $M$ which satisfies the conditions

$$\nabla' F = \nabla' P = 0,\ \ T' \circ P_3 \times P_4 = 0, \tag{4.16}$$

where $T'$ is the torsion of $\nabla'$.

**Definition 4.4.** *The connections* $\nabla$ *and* $\nabla'$*, which satisfy the conditions* (4.14) *and* (4.16) *respectively, will be called the first and the second canonical connection associated to* (*abpc*)*-structure* $(F, P, J)$ *on* $M$.

From the analogous of (4.15) for $\nabla'$ one obtains:

**Proposition 4.4.** *For an* (*abpc*)*-structure* $(F, P, J)$ *on* $M$*, the second canonical connection* $\nabla'$ *may be expressed with the help of the first canonical connection* $\nabla$ *by the relations:*

$$\begin{aligned} &\nabla'_{X_3} X_4 = \nabla_{X_3} X_4 - P_4(T(X_3, X_4)), \\ &\nabla'_{X_4} X_3 = \nabla_{X_4} X_3 - P_3(T(X_4, X_3)), \\ &\nabla'_{X_3} Y_3 = \nabla_{X_3} Y_3 - P_3 \circ F(T(X_3, FY_3)), \\ &\nabla'_{X_4} Y_4 = \nabla_{X_4} Y_4 - P_4 \circ F(T(X_4, FY_4)). \end{aligned} \tag{4.17}$$

From here one obtains

**Proposition 4.5.** *The first and second canonical connection $\nabla$ and $\nabla'$ coincide iff one of the following conditions is satisfied*

$$T \circ P_3 \times P_4 = 0;\;\; T' \circ F_1 \times F_2 = 0. \tag{4.18}$$

**5. Involutivity and integrability.** If $(F, P, J)$ is an (abpc)-structure on $M$, for the Nijenhuis tensor fields of $F$ and $P$, we obtain

(4.19)

$$N_F(X_1, Y_1) = 4F_2[X_1, Y_1],\;\; N_F(X_1, Y_2) = 0, N_F(X_2, Y_2) = 4F_1[X_2, Y_2];$$
$$N_P(X_3, Y_3) = 4P_4[X_3, Y_3], N_P(X_3, Y_4) = 0, N_P(X_4, Y_4) = 4P_3[X_4, Y_4].$$

Then, if $\nabla$ (respectively $\nabla'$) is a connection on $M$, compatible with $F$ (respectively $P$) we get

$$\begin{aligned} &N_F(X_1, Y_1) = -4F_2 \circ T(X_1, Y_1)), N_F(X_1, Y_2) = 0,\\ &N_F(X_2, Y_2) = -4F_1 \circ T(X_2, Y_2);\\ &N_P(X_3, Y_3) = -4P_4 \circ T'(X_3, Y_3), N_P(X_3, Y_4) = 0,\\ &N_P(X_4, Y_4) = -P_4 \circ T'(X_4, Y_4). \end{aligned} \tag{4.20}$$

From these formulas it follows:

**Theorem 4.2.** a) *The eigendistribution $V_i, i = 1, 2, 3, 4$ is involutive iff it is satisfied respectively one of the following conditions for:*

$V_1$: 1) $F_2[X_1, Y_1] = 0$, 2) $N_F(X_1, Y_1) = 0$, 3) $F_2 \circ N_F = 0$,
4) $F_2 \circ T(X_1, Y_1) = 0$.

$V_2$: 1) $F_1[X_2, Y_2] = 0$, 2) $N_F(X_2, Y_2) = 0$, 3) $F_1 \circ N_F = 0$,
4) $F_1 \circ T(X_2, Y_2) = 0$,

$V_3$: 1) $P_4[X_3, Y_3] = 0$, 2) $N_P(X_3, Y_3) = 0$, 3) $P_4 \circ N_P = 0$,
4) $P_4 \circ T'(X_3, y_3) = 0$,

$V_4$: 1) $P_3[X_4, Y_4] = 0$, 2) $N_P(X_4, Y_4) = 0$, 3) $P_3 \circ N_P = 0$,
4) $P_3 \circ T'(X_4, Y_4) = 0$,

*where $T$ (respectively $T'$) is the torsion of a connection $\nabla$ (respectively $\nabla'$) compatible with $F$ (respectively $P$).*

b) *$V_1$ and $V_2$ are simultaneous involutive iff $N_F = 0$ or $T = \overset{1}{T} + \overset{2}{T}$, where $T$ is the torsion of $\nabla$ and $\overset{1}{T}, \overset{2}{T}$ the torsion of the induced connections on $V_1$ and $V_2$ respectively.*

c) *$V_3$ and $V_4$ are simultaneous involutive iff $N_P = 0$, or $T' = \overset{3}{T'} + \overset{4}{T'}$, where $T'$ is the torsion of $\nabla'$ and $\overset{3}{T'}, \overset{4}{T'}$ the torsion of the induced connections on $V_3$ and $V_4$ respectively.*

d) *$V_1, V_2$ and $V_3$ are simultaneous involutive iff is satisfied one of the conditions* a), b), c)*, for each of them.*

e) *$V_1, V_2, V_3$ and $V_4$ are simultaneous involutive iff one of the following conditions is satisfied*

$$N_F = N_P = 0; \;\; T = 0; \;\; T' = 0,$$

*where $T$ and $T'$ are the torsions of the first and second canonical connections.*

**Remark.** The condition d) is very important because an (abpc)-structure $(F, P, J)$, for which all the distributions $V_1, V_2, V_3$ are involutive, is equivalent with a 3-web on $M$ [1.] So, the theory of 3-webs is subordinated to the theory of (abpc)-structures or to anyone of the structures that are equivalent with them.

**Definition 4.5.** *An (abpc)-structure $(F, P, J)$ is integrable iff there exists an atlas on $M$ so that the associated natural local bases are adapted to the structure.*

Concerning the integrability for an (abpc)-structure, from [16] one obtains:

**Proposition 4.5.** An (abpc)-structure $(F, P, J)$ is integrable iff there exists on $M$ a flat $(F, P, J)$-connection.

From here and the uniqueness of the canonical connection one obtains:

**Theorem 4.3.** *The (abpc)-structure $(F, P, J)$ is integrable if the first or the second canonical connections is flat.*

**Definition 4.6.** *A connection* $\nabla$ *on* $M$ *is compatible with the Riemannian* $(mabpc)$*-structure* $(F,P,J,g_1)$ *or is a* $(F,P,J,g_1)$*-connection, iff it satisfies the conditions*

$$\nabla F = \nabla P = \nabla g_1 = 0. \tag{4.21}$$

For such a connection one has also

$$\nabla J = \nabla g_2 = \nabla g_3 = \nabla \omega = 0,\ \nabla F_1 = \nabla F_2 = \nabla P_1 = \nabla P_2 = 0. \tag{4.22}$$

The following theorem holds

**Theorem 4.4.** *Let* $(F,P,J,g_1)$ *be a Riemannian* $(mabpc)$*-structure on* $M$ *and* $\overset{1}{g} = g_1/V_1 \times V_1, \overset{2}{g} = g_1/V_2 \times V_2$*, the induced metrics on* $V_1$ *and* $V_2$*. There exists an unique* $(F,P,J,g_1)$*-connection* $D$ *on* $M$*, which satisfies the conditions:*

$$\overset{i}{D}_X \overset{i}{g} = 0,\ \ \overset{i}{T} = 0,\ \ X \in \mathcal{T}^1(M,V_i),\ \ i = 1,2, \tag{4.23}$$

*where* $\overset{i}{D}$ *is the connection on* $V_i$ *induced by* $D$ *and* $\overset{i}{T}$ *is its torsion.*

*Uniqueness.* Indeed, from Proposition 4.3 it follows that a connection $D$, compatible with the (abpc)-structure $(F,P,J)$, is uniquely determined by $\overset{1}{D}_X, X \in \mathcal{T}^1(M,N_1)$ and $\overset{2}{D}_X, X \in \mathcal{T}^1(M,V_2)$. But, from (4.23) one has

$$\begin{aligned} &X\overset{i}{g}(Y,Z) = \overset{i}{g}(\overset{i}{D}_X Y, Z) + \overset{i}{g}(Y, \overset{i}{D}_X Z),\\ &\overset{i}{D}_X Y - \overset{i}{D}_Y X = F_i[X,Y], X,Y,Z \in \mathcal{T}^1(M,V_i), i = 1,2. \end{aligned} \tag{4.24}$$

By analogously computation with that used in the Riemannian case, [9] we obtain, for any $X,Y,Z \in \mathcal{T}^1(M,V_i), i = 1,2,$
(4.25)

$$\begin{aligned} 2\overset{i}{g}(\overset{i}{D}_X Y, Z) &= X\overset{i}{g}(Y,Z) + Y\overset{i}{g}(Z,X) - Z\overset{i}{g}(X,Y) - \overset{i}{g}(F_i[Y,Z],X)\\ &\quad + \overset{i}{g}(F_i[Z,X],Y) + \overset{i}{g}(F_i[X,Y],Z). \end{aligned}$$

As these formulas determine uniquely $\overset{i}{D}_X, X \in \mathcal{T}^i(M,V_i), i = 1,2,$ the uniqueness of $D$ is proved.

*Existence.* Let $\overset{i}{D}_X, X \in \mathcal{T}^1(M, V_i)$, $i = 1, 2$, be given by (4.25). From Proposition 4.3, it results that these partial connections determine a connection $D$ on $M$, compatible with the (abpc)-structure $(F, P, J)$ and by a simple computation we find that $D$ satisfies also the conditions (4.23) and that $Dg_1 = 0$.

**Definition 4.7.** *The unique connection $D$, given by the* Theorem 4.4, *will be called the first natural connection associated to Riemannian (mabpc)-structure $(F, P, J, g_1)$.*

**Remark.** The first natural connection $D$ satisfies also the condition:

$$\overset{i}{D}\overset{i}{g} = 0, \quad Dg_i = 0, \quad i = 1, 2; \quad D\omega = 0. \tag{4.26}$$

Changing in Theorem 4.4 $F$ with $P$ and $(V_1, V_2)$ with $(V_3, V_4)$ one obtains another connection $D'$ compatible with the Riemannian (mabpc)-structure $(F, P, J, g_1)$, called the second natural connection.

From here it follows:

**Proposition 4.6.** *The first natural connection $D$ coincides with the Levi-Civita connection for one of the metrique $g_i, i = 1, 2, 3$ (and so for all) iff it is torsionless.*

In this case $D$ coincides also with the second natural connection $D'$ and with the first and second canonical connection $\nabla$ and $\nabla'$. Also in this case the structures $F, P, J, \omega$ are integrable and the distributions $V_i, i = 1, 2, 3, 4$ are involutive. Hence we have:

**Theorem 4.8.** *If the first natural connection $D$ for the Riemannian (mabpc)-structure $(F, P, J, g_1)$ is torsionless, then one obtains on $M$:*

a) *two Riemannian local decomposable structures $(F, g_1)$ and $(P, g_1)$, with the associated neutral metrics $g_2$ and $g_3$ respectively,*

b) *two neutral local decomposable structures $(F, g_2)$ and $(P, g_3)$, with the associated Riemannian metric $g_1$,*

c) *two para-Kähler structures $(F, g_3)$ and $(P, g_2)$ with the associated symplectic 2-forms $\omega$ and $-\omega$, respectively,*

d) *a Kähler structure $(J, g_1)$ with the associated symplectic 2-form $\omega$ and*

e) *two anti-Kähler structures* $(J, g_2)$ *and* $(J, g_3)$ *with the associated neutral metrics* $-g_3$ *and* $g_2$*, respectively.*

**5. Example.** Let $N$ be a paracompact and connected manifold, $M = TN$ the total space of the tangent bundle $\pi : TN \to N$ and $VTN = Ker\, T\pi$, the vertical subbundle of $TN$. Denote by $(x^i)$ the local coordinates and by $(\partial_i)$, $(d^i)$, where $\partial_i = \frac{\partial}{\partial x^i}, d^i = dx^i$, the associated local dual bases on $N$. Setting for each 1-form $\alpha \in \mathcal{T}_1(N)$, given locally by $\alpha(x) = \alpha_i(x)d^i$, $\gamma\alpha(z) = \alpha_i(x)y^i$, where $z = (x, y) \in T_xN$, we obtain a class of functions on $TN$, with the property that each vector field $A \in \mathcal{T}^1(TN)$ is uniquely determined by its values on these functions. We extend $\gamma$ to tensor fields $S \in \mathcal{T}_1^1(TN)$ by putting $\gamma S(\gamma\alpha) = \gamma(\alpha \circ S), \forall \alpha \in \mathcal{T}_1(N)$. Let be then $\nabla$ a connection on $N$ and $X \in \mathcal{T}^1(N)$. Setting

$$X^h(\gamma\alpha) = \gamma(\nabla_X \alpha), \ \ X^v(\gamma\alpha) = \alpha(X) \circ \pi, \ \ \forall \alpha \in \mathcal{T}_1(N), \tag{5.1}$$

we obtain two vector fields on $TN$, called respectively, the horizontal and the vertical lift of $X$. Putting then, for each $f \in \mathcal{F}(N), f^h = f^v = f \circ \pi$, one obtains the following useful formulas

$$\begin{aligned} &(fX)^h = f^h X^h, \ \ (fX)^v = f^v X^v, \\ &[X^h, Y^h] = [X, Y]^h - \gamma R_{XY}, \ [X^h, Y^v] = (\nabla_X Y)^v, \ [X^v, Y^v] = 0, \end{aligned} \tag{5.2}$$

where $X, Y \in \mathcal{T}^1(N)$ and $R$ is the curvature of $\nabla$.

Considering then the tensor fields $F, PJ$ given by

$$\begin{aligned} &F(X^h) = X^h, \ F(X^v) = -X^v, \ P(X^h) = X^v, P(X^v) = X^h, \\ &J(X^h) = X^v, J(X^v) = -X^h, \end{aligned} \tag{5.3}$$

for each $X \in \mathcal{T}^1(N)$ it comes out that they satisfy (2.1) and so we have:

**Proposition 5.1.** *Given a connection* $\nabla$ *on* $N$*, the tensor fields* $F, P, J$ *defined by* (5.3) *determine an* (*abpc*)*-structure on the total space* $TN$*.*

The eigendistributions $V_i, i = 1, 2, 3, 4$ associated to $F$ and $P$ are generated respectively by $X^h, X^v, X^h + X^v, X^h - X^v, \forall X \in \mathcal{T}^1(N)$. For the first canonical connection, of the (abpc)-structure $(F, P, J)$,on $TN$, denoted by $D$, we obtain from (4.15) and (5.3)

$$\begin{aligned} &D_{X^h} Y^h = (\nabla_X Y)^h, \ D_{X^h} Y^v = (\nabla_X Y)^v, \\ &D_{X^v} Y^h = D_{X^v} Y^v = 0, X, Y \in \mathcal{T}^1(N) \end{aligned} \tag{5.4}$$

and so it follows:

**Proposition 5.2.** *The first canonical connection of the (abpc)-structure* $(F, P, J)$ *on* $TN$*, given by* (5.3)*, associated to connection* $\nabla$ *on* $N$*, coincides with the diagonal lift of* $\nabla$*, see* [5].

Going further, for nonvanishing components of the torsion and curvature of $D$, one obtains

$$\begin{aligned} &\mathcal{T}(X^h, Y^h) = T(X, Y)^h + \gamma R_{XY}, \mathcal{R}_{X^hY^h} Z^h = (R_{XY}Z)^h, \\ &\mathcal{R}_{X^hY^h} Z^v = (R_{XY}Z)^v, \end{aligned} \tag{5.5}$$

where $T$ and $R$ are the torsion and curvature of $\nabla$.

From (5.5) and Proposition 4.5 it results:

**Proposition 5.3.** *The (abpc)-structure* $(F, P, J)$ *on* $TN$*, associated by* (5.3) *to connection* $\nabla$ *on* $N$*,is integrable iff* $\nabla$ *is a flat connection.*

Let be now $g$ a Riemannian metric on $N$, $\nabla^g$ the Levi-Civita connection of $g$ and $(F, P, J)$ the (abpc)-structure on $TN$ associated to $\nabla^g$. We consider the $(0, 2)$-tensor fields $g^h, g^v, g^{vh}$ and $g^{hv}$ on $TN$ given, for each $X, Y \in \mathcal{T}^1(N)$, by:

$$\begin{aligned} &g^h(X^h, Y^h) = g(X, Y) \circ \pi, \\ &g^h(X^h, Y^v) = g^h(X^v, Y^h) = g^h(X^v, Y^v) = 0, \\ &g^v(X^h, Y^h) = g^v(X^h, Y^v) = g^v(X^v, Y^h) = 0, \\ &g^v(X^v, Y^v) = g(X, Y) \circ \pi, \\ &g^{vh}(X^h, Y^h) = g^{vh}(X^h, Y^v) = g^{vh}(X^v, Y^v) = 0, \\ &g^{vh}(X^v, Y^h) = g(X, Y) \circ \pi, \\ &g^{hv}(X^h, Y^h) = g^{hv}(X^v, Y^h) = g^{hv}(X^v, Y^v) = 0, \\ &g^{hv}(X^h, Y^v) = g(X, Y) \circ \pi. \end{aligned} \tag{5.6}$$

Setting then

$$g_1 = g^h + g^v, \tag{5.7}$$

we obtain a Riemannian metric on $TN$, called the Sasaki metric associated to $g$ which satisfies the conditions

$$g_1 \circ F \times F = g_1 \circ P \times P = g_1 \circ J \times J = g_1. \tag{5.8}$$

From here one obtains:

**Proposition 5.4.** *A Riemannian metric $g$ on $N$ determines by* (5.3), (5.6) *and* (5.7) *a Riemannian* (*mapbc*)*-structure* $(F, P, J, g_1)$ *on* $TN$*, with the associated metrics* $g_2, g_3$ *and 2-form* $\omega$*, given by*

$$g_2 = g^h - g^v, \quad g_3 = g^{vh} + g^{hv}, \quad \omega = g^{vh} - g^{hv}. \tag{5.9}$$

Denoting by $\mathcal{D}$ the first natural connection associated to Riemannian (mabpc)-structure $(F, P, J, g_1)$, given by (5.3) and (5.7), we obtain from (4.25) and Proposition 4.3,

$$\mathcal{D}_{X^h} Y^h = (\nabla^g_X Y)^h, \; \mathcal{D}_{X^h} Y^v = (\nabla^g_X Y)^v, \; \mathcal{D}_{X^v} Y^h = \mathcal{D}_{X^v} Y^v = 0. \tag{5.10}$$

Hence, we have:

**Proposition 5.5.** *The first natural connection of the Riemannian* (*mapbc*)*-structure* $(F, P, J, g_1)$ *on* $TN$*, associated to Riemannian metric* $g$ *on* $N$*, coincide with the diagonal lift for the Levi-Civita connection of the metric* $g$*.*

**Remark.** For a Riemannian metric $g$ on $N$ and its Levi-Civita connection $\nabla^g$, the first natural connection, associated to Riemannian (mabpc)-structure $(F, P, J, g_1)$ coincides with the first canonical connection, associated to (abpc)-structure $(F, P, J)$ on $TN$, determined by $g$.

The nonvanishing components for the first natural connection $\mathcal{D}$ are given by

(5.6)
$$\mathcal{T}(X^h, Y^h) = \gamma R^g_{XY}, \mathcal{R}_{X^h Y^h} Z^h = (R^g_{XY} Z)^h, \; \mathcal{R}_{X^h Y^h} Z^v = (R^g_{XY} Z)^v,$$

where $R^g$ is the curvature of $\nabla^g$. From here it follows:

**Proposition 5.6.** *The torsion and the curvature for the first natural connection* $\mathcal{D}$ *are simultaneously zero, namely iff the curvature of the Levi-Civita connection* $\nabla^g$ *of* $g$ *is zero.*

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