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A PRODUCT BICOMPLEX STRUCTURE ON THE TOTAL SPACE OF A VECTOR BUNDLE

BY

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Abstract. One studies an almost product bicomplex structure on the total space of a vector bundle, obtained by lifting an almost Hermitian structure on the base manifold and one on the bundle, with the help of a linear connection, defined on the bundle. Finally, some important particular cases are analysed.

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1. Introduction. An *almost product bicomplex* (*apbc*)-structure [2] on a manifold M is defined by three $(1,1)$-tensor fields F, J, J' on M, which satisfy the conditions

$$-F^2 = J^2 = J'^2 = F \circ J \circ J' = -I, \quad F \neq \pm I. \tag{1.1}$$

It follows that F is an *almost product* (*ap*)-structure and J, J' are *almost complex* (*ac*)-structures, on M, which are connected by the following relations

$$F \circ J = J \circ F = J', \quad J \circ J' = J' \circ J = -F, \quad J' \circ F = F \circ J' = J. \tag{1.2}$$

A *Riemannian apbc*-structure on a manifold M is a quadruple (F, J, J', G), where (F, J, J') is an *apbc*-structure and G is a Riemannian metric on M, so that

$$G \circ F \times F = G \circ J \times J = G \circ J' \times J' = G, \tag{1.3}$$

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i.e. G is invariant with respect to the automorphisms F, J and J' of TM.

We consider further

$$G' = G \circ I \times F, \quad \Omega = G \circ I \times J, \quad \Omega' = G \circ I \times J' \tag{1.4}$$

and we obtain the following table of compatibilities [2].

$\circ$	F	J	J'
G	$G \circ F \times F = G$ $G \circ I \times F = G'$	$G \circ J \times J = G$ $G \circ I \times J = \Omega$	$G \circ J' \times J' = G$ $G \circ I \times J' = \Omega'$
G'	$G' \circ F \times F = G'$ $G' \circ I \times F = G$	$G' \circ J \times J = G'$ $G' \circ I \times J = \Omega'$	$G' \circ J' \times J' = G'$ $G' \circ I \times J' = \Omega$
Ω	$\Omega \circ F \times F = \Omega$ $\Omega \circ I \times F = \Omega'$	$\Omega \circ J \times J = \Omega$ $\Omega \circ I \times J = -G$	$\Omega \circ J' \times J' = \Omega$ $\Omega \circ I \times J' = -G'$
Ω'	$\Omega' \circ F \times F = \Omega'$ $\Omega' \circ I \times F = \Omega$	$\Omega' \circ J \times J = \Omega'$ $\Omega' \circ I \times J = G'$	$\Omega' \circ J' \times J' = \Omega'$ $\Omega' \circ I \times J' = -G$

From here, we get that to a Riemannian *apbc*-structure on a manifold M there are subordinate the following structures:

1) A Riemannian structure (F, G) with the associated pseudo-Riemannian metric G'.

2) An almost symplectic *ap*-structure (F, Ω) with the associated 2-form Ω'.

3) Two almost Hermitian structures (J, G) and (J', G) with the associated 2-forms Ω and Ω' respectively.

4) Two indefinit Hermitian structures (J, G') and (J', G') with the associated 2-forms Ω' and Ω respectively.

The *apbc*-structures on a manifold were considered by Liberman [6] and more recently by Bonome, Castro, Garcia-Rio, Hervella and Matsushita in the joint paper [1] and by Cruceanu [2].

In this paper, we give an example of an *apbc*-structure on the total space of a vector bundle, obtained by lifting an almost Hermitian structure on the base manifold and one on the bundle, using a linear connection on the bundle.

2. Definitions and notations. Let $\xi = (E, \pi, M)$ be a vector bundle with base manifold M, total space E and projection $\pi : E \to M$. Denote by $(x^i), (y^a), (x^i, y^a)$ the local coordinates on M, ξ, E respectively and by $(\partial_i, d^i), (e_a, e^a), (\partial_i, \partial_a, d^i, d^a)$, the corresponding dual local bases, where $\partial_i = \frac{\partial}{\partial x^i}, d^i = dx^i, \partial_a = \frac{\partial}{\partial y^a}, d^a = dy^a, i,j,k, = 1,2,\dots,n, a,b,c = 1,2,\dots,m, n = \dim M$, $m = \operatorname{rank}\xi$. Denote then by $\mathcal{F}(M)$ the ring of real functions, and by $\mathcal{T}_s^r(M), \mathcal{T}_s^r(\xi)$ the $\mathcal{F}(M)$-module of (r,s)-tensor fields of M and ξ. Next, for each 1-forme α on ξ, given locally by $\alpha(x) = \alpha_a(x)e^a$, we put $\gamma\alpha(z) = \alpha_a(x)y^a$, where $z = (x^i, y^a) \in E_x$ and obtain a class of functions on E, with the property that every vector field on E is uniquely determined by its values on these functions. The mapping γ may be extended to $(1,1)$-tensor fields S on ξ, by putting

$$\gamma S(\gamma\alpha) = \gamma(\alpha \circ S), \;\; \forall \alpha \in \mathcal{T}_1(\xi). \tag{2.1}$$

Locally, if $S(x) = S_b^a(x)e_a \otimes e^b$, then $\gamma S(z) = S_b^a(x)y^b\partial_a$ and so γS is a vector field on E. Then, let D be a linear connection on ξ, X a vector field on M and u a section on ξ. Setting

$$x^h(\gamma\alpha) = \gamma(D_X\alpha), \;\; u^v(\gamma\alpha) = \alpha(u)\circ\pi, \;\; \forall\alpha \in \mathcal{T}_1(\xi), \tag{2.2}$$

one obtain two vector fields on the total space E, called respectively the *horizontal lift* of X and the *vertical lift* of u. One has the useful relations

$$\begin{aligned} &f^v = f^h = f\circ\pi, \;\; (fX)^h = f^hX^h, \;\; (fu)^v = f^vu^v, \\ &[X^h, Y^h] = [X,Y]^h - \gamma R^D_{XY}, \;\; [X^h, u^v] = (D_Xu)^v, \;\; [u^v, v^v] = 0, \end{aligned} \tag{2.3}$$

where $f \in \mathcal{F}(M)$, $X, Y \in \mathcal{T}^1(M)$, $u, v \in \mathcal{T}^1(\xi)$ and R^D is the curvature of D.

Putting now

$$F(X^h) = X^h, \;\; F(u^v) = -u^v, \;\; X \in \mathcal{T}^1(M), \;\; u \in \mathcal{T}^1(\xi), \tag{2.4}$$

we obtain an *ap*-structure on E, whose $+1$ and -1 eigendistributions (subbundles) are respectively the *horizontal distribution* HTE, associated to linear connection D and the *vertical distribution* VTE of the bundle ξ.

For $f \in \mathcal{T}_1^1(M), \varphi \in \mathcal{T}_1^1(\xi)$ and $g \in \mathcal{T}_2(M), \psi \in \mathcal{T}_2(\xi)$, we define the *horizontal* (h) and *vertical* (v) lifts by

$$\begin{aligned} &f^h(X^h) = f(X)^h, \;\; f^h(u^v) = 0; \;\; \varphi^v(X^h) = 0, \;\; \varphi^v(u^v) = \varphi(u)^v; \\ &g^h(X^h, Y^h) = g(X,Y)^h, \;\; g^h(X^h, u^v) = g^h(u^v, X^h) = g^h(u^v, v^v) = 0; \\ &\psi^v(X^h, Y^h) = \psi^v(X^h, u^v) = \psi^v(u^v, X^h) = 0, \; \psi^v(u^v, v^v) = \psi(u,v)^v, \end{aligned} \tag{2.5}$$

for each $X, Y \in \mathcal{T}^1(M)$ and $u, v \in \mathcal{T}^1(\xi)$.

Putting then, for $f \in \mathcal{T}_1^1(M)$ and $\varphi \in \mathcal{T}_1^1(\xi)$,

$$J = f^h + \varphi^v, \quad J' = f^h - \varphi^v \tag{2.6}$$

are remarking that $F = I_{TM}^h - I_\xi^v$, we obtain

$$\begin{aligned} &F^2 = I, \; J^2 = J'^2 = (f^h)^2 + (\varphi^v)^2 \;\; F \circ J = J \circ F = J', \\ &J \circ J' = J' \circ J = (f^h)^2 - (\varphi^v)^2, \;\; J' \circ F = F \circ J' = J. \end{aligned}$$

So, we have

Theorem 2.1. *Given a linear connection D on ξ and two (1.1)-tensor fields f on M and φ on ξ, the triplet (F, J, J') defined by (2.4) and (2.5) determines on E an apbc-structure iff f and φ are ac-structures on M and ξ respectively.*

Now, let g and ψ be $(0, 2)$-tensor fields on M resp. ξ and

$$\omega = g \circ I \times f, \quad \tau = \psi \circ I \times \varphi. \tag{2.7}$$

Setting

$$G = g^h + \psi^v, \;\; G' = g^h - \psi^v, \;\; \Omega = \omega^h + \tau^v, \;\; \Omega' = \omega^h - \tau^v \tag{2.8}$$

one obtains

$$\begin{aligned} &G \circ F \times F = G, \;\; G' \circ F \times F = G', \\ &\Omega \circ F \times F = \Omega, \;\; \Omega' \circ F \times F = \Omega', \\ &G \circ J \times J = G \circ J' \times J' = (g \circ f \times f)^h + (\psi \circ \varphi \times \varphi)^v \\ &G' \circ J \times J = G' \circ J' \times J' = (g \circ f \times f)^h - (\psi \circ \varphi \times \varphi)^v \\ &\Omega \circ J \times J = \Omega \circ J' \times J' = (\omega \circ f \times f)^h + (\tau \circ \varphi \times \varphi)^v \\ &\Omega' \circ J \circ J = \Omega' \circ J' \times J' = (\omega \circ f \times f)^h - (\tau \circ \varphi \times \varphi)^v. \end{aligned} \tag{2.9}$$

Hence we have

Theorem 2.2. *Let D be a linear connection on ξ, f and ψ $(1, 1)$-tensor fields on M resp. ξ, g and ψ $(0, 2)$-tensor fields on M, resp. ξ and $\omega = g \circ I \times f$, $\tau = \psi \circ I \times \varphi$. Then, the quadruple (F, J, J', G), given* *by* (2.4), (2.6) *and* (2.7), *determines on* E *a Riemannian apbc-structure, with the associated metric* G' *and* 2*-forms* Ω, Ω', *iff* (f, g) *and* (φ, ψ) *are almost-Hermitian structures on* M, *resp.* ξ *and* ω, τ *are theirs associated 2-forms.*

Hence the structures $F, J, J', G, G', \Omega, \Omega'$ satisfy all the conditions of compatibility from the table (1.5).

3. The integrability of the structures F, J, J' **and** Ω, Ω'**.** For the Nijenhuis tensor fields of F, J and J' we obtain

(3.1)

$$N_F(X^h, Y^h) = -4\gamma R^D_{XY}, \quad N_F(X^h, u^v) = N_F(u^v, v^v) = 0;$$
$$N_J(X^h, Y^h) = N_f(X, Y)^h - \gamma(R^D_{fXfY} - R^D_{XY} - \varphi \circ (R^D_{fXY} + R^D_{XfY})),$$
$$N_J(X^h, u^v) = (D_{fX}\varphi - \varphi \circ D_X\varphi)(u)^v, \quad N_J(u^v, v^v) = 0;$$
$$N_{J'}(X^h, Y^h) = N_f(X, Y)^h - \gamma(R^D_{fXfY} - R^D_{XY} + \varphi \circ (R^D_{fXY} + R^D_{XfY})),$$
$$N_{J'}(X^h, u^v) = (D_{fX}\varphi + \varphi \circ D_X\varphi)(u)^v, \quad N_{J'}(u^v, v^v) = 0.$$

From here it follows:

Proposition 3.1. 1) *The ap-structure* F *is integrable iff.* $R^D = 0$.

2) *The ac-structure* J *is integrable iff*

$$N_f = 0, \quad R^D_{fXfY} - R^D_{XY} - \varphi \circ (R^D_{fXY} + R^D_{XfY}) = 0, \qquad (3.2)$$
$$D_{fXf} - \varphi \circ D_X\varphi = 0.$$

3) *The ac-structure* J' *is integrable iff*

$$N_f = 0, \quad R^D_{fXfY} - R^D_{XY} + \varphi \circ (R^D_{fXY} + R^D_{XfY}) = 0, \qquad (3.3)$$
$$D_{fX}\varphi + \varphi \circ D_X\varphi = 0.$$

4) The *ac*-structures J and J' are simultaneously integrable iff

$$N_f = 0, \quad R^D_{fXfY} = R^D_{XY}, \quad D\varphi = 0. \qquad (3.4)$$

5) The *apbc*-structure (F, J, J') is integrable iff

$$N_f = 0, \quad D\varphi = 0, \quad R^D = 0. \qquad (3.5)$$

For the exterior differential of Ω and Ω', we obtain

$$\begin{aligned}
&d\Omega(X^h, Y^h, Z^h) = d\omega(X, Y, Z)^v,\\
&3d\Omega(u^v, Y^h, Z^h) = \gamma(\tau(u) \circ R^D_{XY}),\\
&3d\Omega(u^v, v^v, Z^h) = D_Z\tau(u, v)^v, \;\; D\omega(u^v, v^v, w^v) = 0;\\
&d\Omega'(X^h, Y^h, Z^h) = d\omega(X, Y, Z)^v,\\
&3d\Omega'(u^v, Y^h, Z^h) = -\gamma(\tau(u) \circ R^D_{XY}),\\
&3d\Omega'(u^v, v^v, Z^h) = -D_Z\tau(u, v)^v, \;\; D\omega'(u^v, v^v, w^v) = 0.
\end{aligned}$$

Hence, it follows from here

Proposition 3.2. *The almost symplectic structures Ω and Ω' are simultaneously integrable and namely iff*

$$d\omega = 0, \;\; D\tau = 0, \;\; R^D = 0. \tag{3.7}$$

From the Propositions 3.1 and 3.2 it results

Theorem 3.1. *All the structures $F, J, J', \Omega, \Omega'$ are integrable iff*

$$N_f = 0, \;\; d\omega = 0, \;\; D\varphi = 0, \;\; D\psi = 0, \;\; R^D = 0, \tag{3.8}$$

i.e. iff the structure (f, g) on M is Kahlerian, the structure (φ, ψ) on ξ is covariant constant and the connection D on ξ is flat.

In this case we obtains on E:

1) A Riemannian local decompozable structure (F, G), with the associated pseudo-Riemannian local decompozable structure (F, G').

2) A symplectic local product structure (F, Ω), with the associated symplectic structure Ω'.

3) Two Kählerian structures $(J, G), (J', G)$, with the associated symplectic structures Ω, Ω' respectively.

4) Two indefinite Kählerian structures $(J, G'), (J', G')$ with the associated symplectic structures Ω', Ω respectively.

4. Compatible connections with the Riemannian *apbc*-structure (F, J, J', G)

Definition 4.1. *A connection $\mathcal{D}$ on E is compatible with the Riemannian apbc-structure (F, J, J', G) iff*

$$\mathcal{D}F = \mathcal{D}J = \mathcal{D}J' = \mathcal{D}G = 0. \tag{4.1}$$

Then one has also

$$\mathcal{D}G' = \mathcal{D}\Omega = \mathcal{D}\Omega' = 0. \tag{4.2}$$

We are interested in certain particular compatible connections on E, which are related with some connections on M and ξ. To a pair (∇, D) of connections on M and ξ one associates a connection $\mathcal{D}$ on E, called the *diagonal lift* of the pair (∇, D), (see, [4]), given by

$$\mathcal{D}_{X^h}Y^h = (\nabla_X Y)^h, \ \ \mathcal{D}_{X^h}u^v = (D_X u)^v, \ \ \mathcal{D}_{u^v}X^h = \mathcal{D}_{u^v}v^v = 0. \tag{4.3}$$

For the non-vanishing components of the torsion $\mathcal{T}$ and the curvature $\mathcal{R}$ of the connection $\mathcal{D}$, we have

$$\begin{aligned} &\mathcal{T}(X^h, Y^h) = T^{\nabla}(X,Y)^h + \gamma R^D_{XY}, \ \mathcal{R}_{X^hY^h}Z^h = (R^{\nabla}_{XY}Z)^h, \\ &\mathcal{R}_{X^hY^h}u^v = (R^D_{XY}u)^v. \end{aligned} \tag{4.4}$$

For the covariant derivative of the tensors $F, J, J', G, G', \Omega, \Omega'$ one obtains

$$\begin{aligned} \mathcal{D}F = 0;& \\ &\mathcal{D}_{X^h}J = (\nabla_X f)^h + (D_X\varphi)^v, \ \mathcal{D}_{u^v}J = 0; \\ &\mathcal{D}_{X^h}J' = (\nabla_X f)^h - (D_X\varphi)^v, \ \mathcal{D}_{u^v}J' = 0; \\ &\mathcal{D}_{X^h}G = (\nabla_X g)^h + (D_X\psi)^v, \mathcal{D}_{u^v}G = 0; \\ &\mathcal{D}_{X^h}G' = (\nabla_X g)^h - (D_X\psi)^v, \mathcal{D}_{u^v}G' = 0; \\ &\mathcal{D}_{X^h}\Omega = (\nabla_X \omega)^h + (D_X\tau)^v, \mathcal{D}_{u^v}\Omega = 0; \\ &\mathcal{D}_{X^h}\Omega' = (\nabla_X \omega)^h - (D_X\tau)^v, \ \mathcal{D}_{u^v}\Omega' = 0. \end{aligned}$$

From here it follows

Proposition 4.1. If $\mathcal{D}$ is the diagonal lift on E of the pair of connections (∇, D) on M and ξ, then

$$\begin{aligned} &1)\ \mathcal{D}F = 0 \text{ always}, \\ &2)\ \mathcal{D}J = \mathcal{D}J' = 0 \text{ iff } \nabla f = D\varphi = 0, \\ &3)\ \mathcal{D}G = \mathcal{D}G' = 0 \ \text{ iff } \nabla g = D\psi = 0, \\ &4)\ \mathcal{D}\Omega = \mathcal{D}\Omega' = 0 \ \text{ iff } \nabla\omega = D\tau = 0. \end{aligned} \tag{4.6}$$

5) $\mathcal{D}$ is compatible with the Riemannian *apbc*-structure (F, J, J', G) on E, iff ∇ is compatible with the structure (f, g) on M and D is compatible with the structure (φ, ψ) on ξ.

Now, let ∇ be the Levi-Civita connection of g and D a connection on ξ, that is compatible with φ and ψ (see [3]). In this case, we obtain

$$\begin{aligned} \mathcal{D}F = 0;\quad & \mathcal{D}_{X^h} J = \mathcal{D}_{X^h} J' = (\nabla_X f)^h, \\ & \mathcal{D}_{u^v} J = \mathcal{D}_{u^v} J' = 0,\ \mathcal{D}G = \mathcal{D}G' = 0, \\ & \mathcal{D}_{X^h}\Omega = \mathcal{D}_{X^h}\Omega' = (\nabla_X \omega)^h,\ \mathcal{D}_{u^v}\Omega = \mathcal{D}_{u^v}\Omega' = 0; \\ & \mathcal{T}(X^h, Y^h) = \gamma R^D_{XY},\ \mathcal{R}_{X^h Y^h} Z^h = (R^\nabla_{XY} Z)^h, \\ & \mathcal{R}_{X^h Y^h} u^v = (R^D_{XY} u)^v. \end{aligned} \tag{4.7}$$

From here we obtain

Theorem 4.1. *If ∇ is the Levi-Civita connection for g on M and D a connection compatible with φ and ψ on ξ, then the diagonal lift $\mathcal{D}$ of the pair (∇, D) is compatible with the Riemannian apbc-structure (F, J, J', G) iff $\nabla f = 0$, i.e. the structure (f, g) on M is Kahlerian. Further, $\mathcal{D}$ coincide with the Levi-Civita connection of G, i.e. $\mathcal{T} = 0$, iff we have also $R^D = 0$, hence the connection D on ξ is flat.*

In this case, the structures $F, J, J', \Omega, \Omega'$ are integrable and one obtains the results of the Theorem 3.1.

Another important particular case is that in which (f, g) is a Hermitian structure on M, ∇ is the Hermitian connection of the structure (f, g) [5] and D is a connection on ξ, that is compatible with φ and ψ. Then, one has

$$N_f = 0,\ \nabla f = \nabla g = 0,\ T^\nabla(fX, Y) = T^\nabla(X, fY),\ D\varphi = D\psi = 0. \tag{4.8}$$

We want to know, in this case, when (J, G) is a Hermitian structure on E and the diagonal lift $\mathcal{D}$ of the pair (∇, D) is the Hermitian connection for this structure. From (4.8) and Proposition 3.1 it follows $N_J = 0$ iff

$$R^D_{fXfY} - R^D_{XY} - \varphi \circ (R^D_{fXY} + R^D_{XfY}) = 0. \tag{4.9}$$

After that from (4.7) one obtains that the diagonal lift $\mathcal{D}$ of the pair (∇, D) is the Hermitian connection for (J, G) iff

$$R^D_{fXY} = \mathcal{R}^D_{XfY}. \tag{4.10}$$

From (4.9) and (4.10) it follows

$$R^D_{fXY} = \varphi \circ R^D_{XY}. \tag{4.11}$$

Similarly, considering the same problem for the structure (J', G), we obtain

$$R^D_{fXY} = -\varphi \circ R^D_{XY}. \tag{4.12}$$

Finally, we have

Theorem 4.2. *If ∇ is the Hermitian connection for a Hermitian structure (f, g) on M, D a connection on ξ compatible with a Hermitian structure (φ, ψ) on ξ and (F, J, J', G) is the Riemannian apbc-structure on E, obtained by lifting (f, g) and (φ, ψ) with the help of D, then:*

1) *The structure (J, G), (resp. (J', G)) on E is Hermitian and the diagonal lift $\mathcal{D}$ of the pair $(\nabla, \mathcal{D})$ is the Hermitian connection for this structure, iff $R^D_{fXY} = \varphi \circ R^D_{XY}$, (resp. $R^D_{fXY} = -\varphi \circ R^D_{XY}$).*

2) *The structures (J, G) and (J', G) on E are simultaneously Hermitian and the diagonal lift of the pair (∇, D) is their Hermitian connection iff $R^D = 0$.*

In the last case, $\mathcal{T} = 0$ and so $\mathcal{D}$ coincides with the Levi-Civita connection of G. Hence the structures $F, J, J', \Omega, \Omega'$ are integrable and so one obtains again the results of the Theorem 3.1.

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