32 On certain lifts in the tangent bundle

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1. Introduction. Continuing the study from [5], concerning some lifts for tensor fields and linear connections on a vector bundle, in this paper we shall deal with the particular case of the lifts on the tangent bundle.

The total space of the tangent bundle is a manifold naturally endowed with a very rich geometrical structure and that is why it presents a special interest for Differential Geometry, Analytical Mechanics and Theoretical Physics. See the recent monograph [12] by R. Miron and M. Anastasiei and the references therein.

Starting from a natural and unitary point of view, we obtain new definitions, different from those introduced by K. Yano and S. Ishihara [15], for the vertical and horizontal lifts and we give simple geometrical characterizations for the considered lifts.

1. d–Tensor fields and certain lifts. We shall work in the category of C^∞–manifold. Let M be a connected and paracompact m–dimensional manifold, $\mathcal{F}(M)$ the ring of real functions, $\mathcal{T}^p_q(M)$ the $\mathcal{F}(M)$-module of (p,q)–tensor fields and $\mathcal{T}(M)$ the $\mathcal{F}(M)$–bigraded tensor algebra of M. Let be then (TM, π, M) the tangent bundle of M, $VTM = \operatorname{Ker} T\pi$ the vertical subbundle of $T(TM)$ and $V^\perp TM = \operatorname{Im} T^*\pi$ the subbundle of $T^*(TM)$, dual orthogonal to VTM. Denote by WTM the quotient bundle of $T(TM)$ by VTM and by $W^\perp TM$ the quotient bundle of $T^*(TM)$ by $V^\perp TM$. We obtain the following short exact sequences of vector bundles over the manifold TM

$$0 \longrightarrow VTM \xrightarrow{\;i\;} T(TM) \xrightarrow{\;p\;} WTM \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow V^\perp TM \xrightarrow{\;j\;} T^*(TM) \xrightarrow{\;q\;} W^\perp TM \longrightarrow 0, \tag{2}$$

where i, j and p, q are the canonical injections and projections, respectively. Let be, in the local chart (U, φ) and $(\pi^{-1}(U), \Phi)$ on M and TM, the local coordinates $(x^i), (x^i, y^i)$, respectively and the pairs of corresponding dual bases $(\partial_i; d^i), (\partial_i, \dot{\partial}_i; d^i, \dot{d}^i)$, where $\partial_i = \partial/\partial x^i$, $\dot{\partial}_i = \partial/\partial y^i$, $d^i = dx^i$, $\dot{d}^i = dy^i$, $i, j, k = 1, 2, ..., m$. For $VTM, V^\perp TM, WTM$ and $W^\perp TM$, we obtain, respectively, the natural bases $(\dot{\partial}_i)$, (d^i), $(\bar{\partial}_i = p(\partial_i))$, $(\bar{\dot{d}}^i = q(\dot{d}^i))$. The exact sequences (1) and (2) suggest us to consider the following natural class of "tensor fields" on the manifold TM.

Definition 1.1. A *distinguished*, or shortly, a *d–tensor field* of type (p, q, r, s), on the tangent manifold TM, is a section T of the vector bundle $\otimes^p WTM \otimes^r VTM \otimes^q V^\perp TM \otimes^s W^\perp TM$ over TM.

The local expression for such a tensor field is

$$T(z) = T^{i_1\dots i_p k_1\dots k_r}_{j_1\dots j_q \ell_1\dots \ell_s}(x,y)\bar{\partial}_{i_1} \otimes \cdots \dot{\partial}_{k_1} \otimes \cdots d^{j_1} \otimes \cdots \dot{\bar{d}}^{\ell_1} \otimes \cdots \otimes \dot{\bar{d}}^{\ell_s}. \tag{3}$$

We shall denote by $\bar{\mathcal{T}}^{p,r}_{q,s}(TM)$ and $\bar{\mathcal{T}}(TM)$, the $\mathcal{F}(TM)$-module of d-tensor fields of type (p,q,r,s) and the corresponding fourgraded tensor algebra on TM. The coordinates of d-tensor fields of types $(0,0,p,q)$, $(p,q,0,0)$, $(0,q,p,0)$ and $(p,0,0,q)$ have the same law of transformation as those of tensor fields of type (p,q) on M and so we can consider the following lifts

Definition 1.2. The $vw^{\perp}$, $wv^{\perp}$, $vv^{\perp}$ and $ww^{\perp}$-lift for a tensor field $t \in \mathcal{T}^p_q(M)$, given by

$$t = t^{i_1\dots i_p}_{j_1\dots j_q}(x)\partial_{i_1} \otimes \cdots d^{j_1} \otimes \cdots \otimes d^{j_q}, \tag{4}$$

is the d-tensor field T of type $(0,0,p,q)$, $(p,q,0,0)$, $(0,q,p,0)$ and $(p,0,0,q)$ respectively on TM, given by (3), where

$$T^{i_1\dots i_p}_{j_1\dots j_q}(x,y) = t^{i_1\dots i_p}_{j_1\dots j_q}(x). \tag{5}$$

The ring $\mathcal{F}(M)$ being isomorphic with the subring $\bar{\mathcal{F}}(M) = \{f \circ \pi \mid f \in \mathcal{F}(M)\}$ of $\mathcal{F}(TM)$, it may be considered as a subring of $\mathcal{F}(TM)$. So the previous lifts give four embeddings of the bigraded tensor algebra $\mathcal{T}(M)$ in the fourgraded d-tensor algebra $\bar{\mathcal{T}}(M)$. We remark that the bigraded subalgebra of d-tensor fields on TM of type $(0,q,p,0)$, $p,q \in \mathbb{N}$ is also a subalgebra of the tensor algebra $\mathcal{T}(TM)$. So, by the $vv^{\perp}$-lift, the tensor algebra $\mathcal{T}(M)$ is embedded naturally in the tensor algebra $\mathcal{T}(TM)$. The d-tensor fields of type $(0,q,p,0)$ on TM were considered in many papers with different names: M-tensors [13],[12], Finsler tensors [11], d-tensors [12], semibasic tensors [3] etc.

For example, if I is the identical automorphism of the bundle TM, then $\sigma = (I)^{vv^{\perp}}$ is the *natural almost tangent* structure on the manifold TM with the local expression $\sigma = \dot{\partial}_i \otimes d^i$.

Setting for each 1–form $\omega \in \mathcal{T}_1(M)$, given locally by $\omega(x) = \omega_i(x)d^i$,

$$\gamma(\omega)(z) = \omega_i(x)y^i, \tag{7}$$

where $z = (x,y)$, we obtain a class of functions on TM with the following important property. For two vector fields A and B on TM we have $A = B$ if and only if $A(\gamma\omega) = B(\gamma\omega)$, for each $\omega \in \mathcal{T}_1(M)$. The operator γ may be extended to tensor fields $t \in \mathcal{T}^p_{1+q}(M)$ by

$$\gamma(t^{i_1\dots i_p}_{jj_1\dots j_q}\partial_{i_1} \otimes \cdots \otimes d^j \otimes d^{j_1} \otimes \cdots \otimes d^{j_q})(z) = y^j t^{i_1\dots i_p}_{jj_1\dots j_q}(x)\dot{\partial}_{i_1} \otimes \cdots \otimes \dot{\partial}_{i_p} \otimes d^{j_1} \otimes \cdots \otimes d^{j_q}. \tag{8}$$

In particular, for $t = I$, we get the *canonical* (or *Liouville*) vector field $K = \gamma(I)$ on the manifold TM, given by

$$K(\gamma\omega) = \gamma\omega, \quad \forall\, \omega \in \mathcal{T}_1(M), \tag{9}$$

with the local expression $K(z) = y^i\dot{\partial}_i$.

Definition 1.3. A *vertical* vector field on the manifold TM is a section of the vertical subbundle VTM.

So a vertical vector field is a d-tensor field of type $(0,0,1,0)$ and it has the local expression $A(z) = A^i(x,y)\dot{\partial}_i$.

Definition 1.4. The *vertical* lift for a vector field $X \in \mathcal{T}^1(M)$ is the vector field X^v on TM given by

$$X^v(\gamma\omega) = \omega(X) \circ \pi, \quad \forall\, \omega \in \mathcal{T}_1(M). \tag{10}$$

Locally, if $X = X^i(x)\partial_i$, then $X^v = X^i(x)\dot{\partial}_i$. For $X = \partial_i$ we obtain

(11) $$(\partial_i)^v = \dot{\partial}_i, \quad i = 1, 2, ..., m.$$

Hence, for a vector field on M, the vertical lift coincides with the $vv^\perp$-lift. We remark the properties

(12) $$[X^v, Y^v] = 0, \ \mathcal{L}_K X^v = -X^v, \ \ \forall X, Y \in \mathcal{T}^1(M),$$

where $\mathcal{L}$ is the Lie derivation and K the canonical vector field.

Definition 1.5. A *horizontal* 1-form on the manifold TM is a 1-form which vanishes on every vertical vector field.

Hence, a horizontal 1-form on TM is a section in the subbundle $V^\perp TM$ and it has the local expression $\alpha(z) = \alpha_i(x, y)d^i$.

Definition 1.6. The *horizontal* lift of a 1-form $\omega \in \mathcal{T}_1(M)$ is the 1-form on TM given by

(13) $$\omega^h = T^*\pi(\omega).$$

If $\omega = \omega_i(x)d^i$, then $\omega^h(z) = \omega_i(x)d^i$. For $\omega = d^i$ we get

(14) $$(d^i)^h = d^i, \quad i = 1, 2, ..., m.$$

Hence for a 1-form on M, the horizontal lift coincides with the $vv^\perp$-lift.

Remark 1.1. The horizontal 1-forms and the horizontal lift for 1-forms coincide respectively with the vertical 1-forms and the vertical lifts for 1-forms considered by K. Yano and S. Ishihara [15].

2. Normalization for the vertical foliation. TM being a manifold endowed with the vertical foliation, it is convenient for its study to consider a *normalization* of the foliation, that is, a distribution on TM supplementary to the vertical one. Such a distribution will be called *horizontal* distribution and denoted by HTM. With HTM will be denoted also the corresponding subbundle of $T(TM)$ and we shall call it the *horizontal* subbundle. A normalization can be defined, for example, by a right splitting of the exact sequence (1), that is, a morphism $N : WTM \longrightarrow T(TM)$ so that $p \circ N = I_{WTM}$. Then putting $HTM = N((WTM)$, we obtain an embedding of WTM, as a supplementary subbundle for VTM. Setting locally $\delta_i = N(\bar{\partial}_i)$ we obtain

(15) $$\delta_i = \partial_i - N_i^j(x, y)\dot{\partial}_j, \quad i = 1, 2, ..., m$$

and it follows that (δ_i), $i = 1, 2, ..., m$ is a local basis for HTM. The splitting N is also called *nonlinear* connection. The name is justified by the fact that if we have a linear connection ∇ on M, then putting locally

(16) $$\nabla_{\partial_i}\partial_k = \Gamma_{ik}^j(x)\partial_j \ \text{ and } \ N_i^j(x, y) = \Gamma_{ik}^j(x)y^k,$$

the relations (15) give a normalization N on TM. But generally, for a normalization N, the local functions $N_i^j(x, y)$ are not linear in y.

Definition 2.1. A *horizontal* vector field on the manifold TM, with respect to normalization N, is a section on the horizontal subbundle HTM. Locally, a horizontal vector field is given by $A(z) = A^i(x, y)\delta_i$.

Definition 2.2. The *horizontal* lift of a vector field $X \in \mathcal{T}^1(M)$ is the horizontal vector field X^h on TM which satisfies the condition $T\pi(X^h) = X$. Locally, if $X = X^i(x)\partial_i$, then $X^h(z) = X^i(x)\delta_i$. For $X = \partial_i$, we obtain

$$(\partial_i)^h = \delta_i, \quad i = 1, 2, ..., m. \tag{17}$$

It is not difficult to prove

Proposition 2.1. *If N is the normalization defined by a linear connection ∇ on M, then the horizontal lift X^h, for $X \in \mathcal{T}^1(M)$, is characterized by*

$$X^h(\gamma\omega) = \gamma(\nabla_X \omega), \quad \forall \omega \in \mathcal{T}_1(M). \tag{18}$$

Proposition 2.2. *A normalization N on TM is induced by a linear connection ∇ on M if and only if*

$$\mathcal{L}_K X^h = 0, \quad \forall X \in \mathcal{T}^1(M). \tag{19}$$

Proof. We obtain locally

$$\mathcal{L}_K X^h = \left(N_k^i - \frac{\partial N_k^i}{\partial y^j} y^j \right) X^k \dot{\partial}_i$$

and N_k^i being of class C^∞ on TM, the condition (19) is equivalent with $(16)_2$. In this case one has

$$[X^h, Y^h] = [X, Y]^h - \gamma R_{XY}, \ \forall X, Y \in \mathcal{T}^1(M), \tag{20}$$

where R is the curvature of the linear connection ∇.

Remark 2.1. In [15], for a linear connection ∇ on M, given locally by $(16)_1$, the horizontal lift, for a vector field $X \in \mathcal{T}^1(M)$, is defined by $X^h(z) = X^i(x)(\partial_i - \Gamma_{ki}^j(x) y^k \dot{\partial}_j)$. That is, in [15], the horizontal lift of X, with respect to ∇, coincides with the horizontal lift of X given by the Definition 2.1, with respect to the connection transposed to ∇, i.e. ${}^t\nabla = \nabla - T^\nabla$, where T^∇ is the torsion of ∇. Evidently, the definition in [15] complicates the things in many questions concerning the horizontal lift. This deficience was deleted in [6], but the old definition is still used by certain authors.

Now, we obtain, for the horizontal lift of a 1-form $\omega \in \mathcal{T}_1(M)$, the following characterization.

Proposition 2.3. *The horizontal lift for the 1-form ω on M, is the 1-form ω^h on TM given by*

$$\omega^h(X^v) = 0, \ \omega^h(X^h) = \omega(X) \circ \pi, \ \forall X \in \mathcal{T}^1(M). \tag{21}$$

Definition 2.3. A *vertical* 1-form on TM is a 1-form which vanishes on every horizontal vector field.

Hence, a vertical 1-form is a section in the subbundle $H^\perp TM \subset T^*(TM)$, the orthogonal dual of HTM. Locally, for such a 1-form one has $\alpha(z) = \alpha_i(x, y)(\dot{d}^i + N_j^i(x, y) d^j)$.

Definition 2.4. The *vertical* lift of a 1-form ω on M, with respect to a normalization N, is the 1-form ω^v on TM given by

$$\omega^v(X^v) = \omega(X) \circ \pi, \ \omega^v(X^h) = 0, \ \ \forall X \in \mathcal{T}^1(M). \tag{22}$$

If $\omega=\omega_i(x)d^i$, then $\omega^v(z)=\omega_i(x)(\dot{d}^i+N^i_j(x,y)d^j)$. For $\omega=d^i$ we obtain

$$\dot{\delta}^i=(d^i)^v=\dot{d}^i+N^i_jd^j,\quad i=1,2,...,m. \tag{23}$$

Hence the 1-forms $(\dot{\delta}^i), i=1,2,...,m$ generate locally the subbundle $H^{\perp}TM$.

Remark 2.2. The vertical 1-forms and the vertical lifts for 1-forms coincide with the horizontal 1-forms and the horizontal lifts for 1-forms considered by K. Yano and S. Ishihara [15].

From the previous considerations it results that the following systems of local sections $(\delta_i,\dot{\partial}_i)$ and $(d^i,\dot{\delta}^i)$ represent the dual bases adapted to the normalization N and the natural charts on TM.

These bases are very convenient in the study of TM. From the transformation laws of natural and adapted bases one obtains isomorphisms between VTM and WTM with HTM and between $V^{\perp}TM$ and $W^{\perp}TM$ with $H^{\perp}TM$, which evidently depend on the normalization N.

3. N-decomposable tensor fields and N-lifts.

A normalization N of the vertical foliation determines a direct sum decomposition of the bundles $T(TM)$ and $T^*(TM)$,

$$T(TM)=HTM\oplus VTM,\ T^*(TM)=V^{\perp}TM\oplus H^{\perp}TM. \tag{24}$$

Denoting by H and V the horizontal and the vertical projectors, associated to these decompositions, we obtain for $A\in\mathcal{T}^1(TM)$ and $\alpha\in\mathcal{T}_1(TM)$

$$A=HA+VA,\ \alpha=H\alpha+V\alpha=\alpha\circ H+\alpha\circ V. \tag{25}$$

From (25) and the Definitions 2.3 and 1.5 it follows that the duals $(VTM)^*$ and $(HTM)^*$ are isomorphic with $H^{\perp}TM$ and $V^{\perp}TM$ respectively.

Definition 3.1. A N-*decomposable* tensor field of type (p,q,r,s) on the manifold TM, with respect to the normalization N, is a section of the vector bundle $\otimes^pHTM\otimes^rVTM\otimes^qV^{\perp}TM\otimes^s H^{\perp}TM$.

We denote by $\mathcal{T}^{p,r}_{q,s}(TM,N)$ and $\mathcal{T}(TM,N)$ the $\mathcal{F}(TM)$-module of N-decomposable tensor fields of type (p,q,r,s) and the corresponding fourgraded tensor algebra on TM. Considering a tensor field $\tilde{T}\in\mathcal{T}^{p+r}_{q+s}(TM)$ as a $\mathcal{F}(TM)$-multilinear mapping $\tilde{T}:\mathcal{T}_1(TM)^{p+r}\times\mathcal{T}^1(TM)^{q+s}\longrightarrow\mathcal{F}(TM)$, it follows

Proposition 3.1. *A tensor field* $\tilde{T}\in\mathcal{T}^{p+r}_{q+s}(TM)$ *is* N*-decomposable of type* (p,q,r,s) *if and only if*

$$\tilde{T}=\tilde{T}\circ(H^p\times V^r\times H^q\times V^s). \tag{26}$$

Such a tensor field has the local expression in adapted bases

$$\tilde{T}(z)=\tilde{T}^{i_1...i_pk_1...k_r}_{j_1...j_q\ell_1...\ell_s}(x,y)\delta_{i_1}\otimes\cdots\dot{\partial}_{k_1}\otimes\cdots d^{j_1}\otimes\cdots\dot{\delta}^{\ell_1}\otimes\cdots\otimes\dot{\delta}^{\ell_s}. \tag{27}$$

From (24) and (26) it results that each tensor field $T\in\mathcal{T}^i_j(TM)$ may be decomposed in 2^{i+j} N-decomposable tensor fields of type (p,q,r,s) with $p+r=i$, $q+s=j$. Therefore we obtain for each $i,j\in\mathbb{N}^*$,

$$\mathcal{T}^i_j(TM)=\bigoplus_{\substack{p+r=i\\ q+s=j}}\mathcal{T}^{p,r}_{q,s}(TM,N)$$

and so, the bigraded algebra $\mathcal{T}(TM)$ can be replaced by the fourgraded algebra $\mathcal{T}(TM, N)$.

Definition 3.2. The N-*lift* with respect to normalization N, of a d-tensor field T of type (p, q, r, s) on TM, given by (3), is the N-decomposable tensor field $\tilde{T}$ of the same type, given by (27), where

$$\tilde{T}^{i_1...i_pk_1...k_r}_{j_1...j_q\ell_1...\ell_s}(x, y) = T^{i_1...i_pk_1...k_r}_{j_1...j_q\ell_1...\ell_s}(x, y). \tag{28}$$

Evidently, the N-lift is an isomorphism between the fourgraded algebras $\bar{\mathcal{T}}(TM)$ and $\mathcal{T}(TM, N)$.

In the following table we consider certain important classes of N-decomposable tensor fields of type (p, q, r, s) on the manifold TM.

(29)

Name	Characterization	Local expression
Vertical	$p=q=0$	$T=T^{k_1...k_r}_{\ell_1...\ell_s}\dot{\partial}_{k_1}\otimes\cdots\otimes\dot{\partial}_{k_r}\otimes\dot{\delta}^{\ell_1}\otimes\cdots\otimes\dot{\delta}^{\ell_s}$
Horizontal	$r=s=0$	$T=T^{i_1...i_p}_{j_1...j_q}\delta_{i_1}\otimes\cdots\otimes\delta_{i_p}\otimes d^{j_1}\otimes\cdots\otimes d^{j_q}$
Vertical–horizontal	$p=s=0$	$T=T^{k_1...k_r}_{j_1...j_q}\dot{\partial}_{k_1}\otimes\cdots\otimes\dot{\partial}_{k_r}\otimes d^{j_1}\otimes\cdots\otimes d^{j_q}$
Horizontal–vertical	$p=s=0$	$T=T^{i_1...i_p}_{\ell_1...\ell_s}\delta_{i_1}\otimes\cdots\otimes\delta_{i_p}\otimes\dot{\delta}^{\ell_1}\otimes\cdots\otimes\dot{\delta}^{\ell_s}$

These tensor fields determine four bigraded subalgebras in the algebra $\mathcal{T}(TM, N)$, which will be called respectively: *vertical* $(V\mathcal{T}(TM, N))$, *horizontal* $(H\mathcal{T}(TM, N))$, *vertical-horizontal* $(VH\mathcal{T}(TM, N))$ and *horizontal-vertical* $(HV\mathcal{T}(TM, N))$ subalgebras. We remark that the vertical, horizontal and horizontal–vertical subalgebras depend on the normalization N. The vertical–horizontal subalgebra is independent on the normalization and coincides with the subalgebra of $\mathcal{T}(TM)$ formed by the d-tensor fields of type $(0, q, r, 0)$, $q, r \in \mathbb{N}$. Since $(VTM)^*$ is naturally isomorphic with $H^{\perp}TM$, it follows that the N-lift determines an isomorphism between the tensor algebra of VTM and the subalgebra $V\mathcal{T}(TM, N)$ of the vertical tensor fields on TM, with respect to N. Then, $(HTM)^*$ being isomorphic with $V^{\perp}TM$, the N-lift determines an isomorphism between the tensor algebra of HTM and the subalgebra $H\mathcal{T}(TM, N)$ of the horizontal tensor fields on TM, with respect to the normalization N.

Definition 3.3. The *vertical* (v), *horizontal* (h), *vertical–horizontal* (vh) and *horizontal–vertical* (hv)-lifts of the module $\mathcal{T}^p_q(M)$, with respect to the normalization N on TM, are the $vw^{\perp}$, $wv^{\perp}$, $vv^{\perp}$ and $ww^{\perp}$-lifts composed respectively, with the N-lift, that is:

$$v = N \circ vw^{\perp},\ h = N \circ wv^{\perp},\ vh = N \circ vv^{\perp},\ hv = N \circ ww^{\perp}. \tag{30}$$

Proposition 3.2. *The vertical, horizontal, vertical–horizontal and horizontal–vertical lifts for a tensor field* $t \in \mathcal{T}^p_q(M)$ *are respectively the vertical, horizontal, vertical–horizontal and horizontal–vertical tensor fields on* TM, *denoted by* t^v, t^h, t^{vh}, t^{hv} *and given by*

$$\begin{aligned} t^v(\omega_1^v, ..., \omega_p^v, X_1^v, ..., X_q^v) &= t(\omega_1, ..., \omega_p, X_1, ..., X_q) \circ \pi, \\ t^h(\omega_1^h, ..., \omega_p^h, X_1^h, ..., X_q^h) &= t(\omega_1, ..., \omega_p, X_1, ..., X_q) \circ \pi, \\ t^{vh}(\omega_1^v, ..., \omega_p^v, X_1^h, ..., X_q^h) &= t(\omega_1, ..., \omega_p, X_1, ..., X_q) \circ \pi, \\ t^{hv}(\omega_1^h, ..., \omega_p^h, X_1^v, ..., X_q^v) &= t(\omega_1, ..., \omega_p, X_1, ..., X_q) \circ \pi, \end{aligned} \tag{31}$$

for every $\omega_i \in \mathcal{T}_1(M)$ *and* $X_j \in \mathcal{T}^1(M)$.

If the normalization N is defined by a linear connection ∇ on M, taking into account that $\mathcal{L}_K X^v = -X^v$ and $\mathcal{L}_K X^h = 0$, we obtain for these lifts the following characterizations.

Proposition 3.3. *A tensor field $T \in \mathcal{T}_q^p(TM)$ is the vertical, horizontal, vertical–horizontal or horizontal–vertical lift of a tensor field $t \in \mathcal{T}_q^p(M)$, with respect to normalization N defined by a linear connection ∇ on M, if and only if it is a vertical, horizontal, vertical–horizontal or horizontal–vertical tensor field respectively, which satisfies the corresponding condition*

$$\mathcal{L}_K T = (q-p)T;\ \mathcal{L}_K T = 0;\ \mathcal{L}_K T = -pT;\ \mathcal{L}_K T = qT. \tag{32}$$

4. Structures associated to a normalization on TM. Let N be a normalization of the vertical foliation on the tangent manifold TM and I the identical automorphism of the tangent bundle TM. Using the v, h, vh and hv-lifts associated to N, we obtain the well known [2], [4], [7]-[15] and very important tensor fields of type $(1,1)$ on TM:

$$\begin{aligned} &V = I^v,\ H = I^h,\ \sigma = I^{vh},\ \tau = I^{hv},\ F = I^h - I^v, \\ &P = I^{hv} + I^{vh},\ J = I^{hv} - I^{vh}. \end{aligned} \tag{33}$$

From (31) it results, for these tensor fields, the characterizations:

$$\begin{aligned} &V(X^v)=X^v,\ V(X^h)=0;\ H(X^v)=0,\ H(X^h)=X^h;\ \sigma(X^v)=0, \\ &\sigma(X^h)=X^v;\ \tau(X^v)=X^h,\ \tau(X^h)=0;\ F(X^h)=X^h,\ F(X^v)=-X^v; \\ &P(X^v)=X^h,\ P(X^h)=X^v;\ J(X^h)=-X^v,\ J(X^v)=X^h. \end{aligned} \tag{34}$$

For the composition of the morphisms defined by these tensor fields on the bundle TM, we obtain the table

(35)

$\circ$	V	H	σ	τ	F	P	J
V	V	0	σ	0	$-V$	σ	$-\sigma$
H	0	H	0	τ	H	τ	τ
σ	0	σ	0	V	σ	V	V
τ	τ	0	H	0	$-\tau$	H	$-H$
F	$-V$	H	$-\sigma$	τ	I	J	P
P	τ	σ	H	V	$-J$	I	$-F$
J	τ	$-\sigma$	H	$-V$	$-P$	F	$-I$

So, V and H are the vertical and horizontal projectors, σ is the natural tangent structure, τ is another almost tangent structure (the second), F and P are two almost paracomplex structures [4] (the first and the second), and J is an almost complex structure on TM. Excepting σ, all these structures depend on the normalization N. The tensor fields, F, P and J determine together an almost antiquaternionic structure on TM, associated to N.

The normalization N can be defined by a right or left splitting of the sequence (1) or (2), but it can be also defined by one of the tensor fields V, H, F, P, J, τ on TM. Using the characterizations (34) and computing the Lie derivative of these tensor fields, with respect to canonical vector field K, we can see when the normalization N is induced by a linear connection ∇ on M. After some simple calculations we obtain

Proposition 4.1. *Let be σ the natural tangent structure and K the canonical vector field on the tangent manifold TM. Then:*

(i) *A normalization N on TM can be defined by one of the tensor fields $V, H, F, P, J, \tau \in \mathcal{T}^1_1(TM)$, which satisfy respectively the conditions:*

$$(36)\qquad \begin{array}{l} V\circ\sigma=\sigma,\ \sigma\circ V=0;\ H\circ\sigma=0,\ \sigma\circ H=\sigma;\ F\circ\sigma=-\sigma,\\ \sigma\circ F=\sigma;\ P^2=I,\ P\circ\sigma+\sigma\circ P=I;\\ J^2=-I,\ J\circ\sigma+\sigma\circ J=I;\ \tau^2=0,\ \tau\circ\sigma+\sigma\circ\tau=I. \end{array}$$

(ii) *The horizontal distribution is given respectively by*

$$(37)\qquad \begin{array}{l} \operatorname{Ker} V;\ \operatorname{Im} H;\ F^+=\{A\in\mathcal{T}^1(TM)|F(A)=A\};\\ P(VTM);\ J(VTM);\ \operatorname{Ker}\tau=\operatorname{Im}\tau. \end{array}$$

(iii) *The normalization N is induced by a linear connection on M if and only if one has, respectively*

$$(38)\qquad \mathcal{L}_K V=0;\ \mathcal{L}_K H=0;\ \mathcal{L}_K F=0;\ \mathcal{L}_K P=J;\ \mathcal{L}_K J=P;\ \mathcal{L}_K\tau=\tau.$$

Let φ be a tensor field of type $(1,1)$ on TM. Setting

$$\varphi f=f,\ \varphi A=\varphi(A),\ \varphi\alpha=\alpha\circ\varphi, \varphi T(\alpha_1,...,A_1,...)=T(\varphi\alpha_1,...,\varphi A_1,...)$$

for $f\in\mathcal{F}(TM)$, $A_i\in\mathcal{T}^1(TM)$, $\alpha_j\in\mathcal{T}_1(TM)$, $T\in\mathcal{T}^p_q(TM)$, it results that φ defines an endomorphism of the algebra $\mathcal{T}(TM)$. Considering then φ equal with one of the tensor fields V, H, σ, τ, P, J and taking into account the relations (34), we obtain

Proposition 4.2. *The subalgebras $V\mathcal{T}(TM,N)$, $H\mathcal{T}(TM,N)$, $VH\mathcal{T}(TM,N)$ and $HV\mathcal{T}(TM,N)$ have the following properties:*

$$(39)\qquad \begin{array}{ll} V\mathcal{T}(TM,N)=\operatorname{Im} V=\operatorname{Ker} H, & H\mathcal{T}(TM,N)=\operatorname{Im} H=\operatorname{Ker} V,\\ VH\mathcal{T}(TM,N)=\operatorname{Im}\sigma=\operatorname{Ker}\sigma, & HV\mathcal{T}(TM,N)=\operatorname{Im}\tau=\operatorname{Ker}\tau,\\ P(V\mathcal{T}(TM,N))=H\mathcal{T}(TM,N), & P(VH\mathcal{T}(TM,N))=HV\mathcal{T}(TM,N),\\ J(V\mathcal{T}(TM,N))=H\mathcal{T}(TM,N), & J(VH\mathcal{T}(TM,N))=HV\mathcal{T}(TM,N). \end{array}$$

Remark 4.1. This proposition gives a new justification for the definition adopted by us for the vertical and horizontal tensor fields on TM and for the corresponding lifts of tensor fields on M. The definitions given by K. Yano and S. Ishihara in [15], for the vertical and the horizontal lifts, are not justified, firstly, because the corresponding tensor fields are not generally, vertical or horizontal respectively. Actually the vertical lift of K. Yano and S. Ishihara coincides with our vh-lift and it is independent on the normalization. After that, the horizontal lift of K. Yano and S. Ishihara [15] is an artificial and complicated construction. It has been used in the form of K. Yano and S. Ishihara or easily modified (as we have seen for vector fields), but only for some particular types of tensor fields for which it may be expressed simply with the lifts considered by us. The complet lift, defined by K. Yano and S. Kobayashi [14] is also complicated, but it is a natural and useful construction which is independent of the normalization and is strongly related with our vh-lift.

5. Derivation laws in the algebra of d-tensor fields.

Definition 5.1. A *d–connection* on the manifold TM, with respect to a normalization N, is a linear connection on TM which induces a law of derivation in the algebra of d-tensor fields $\bar{\mathcal{T}}(TM)$.

Proposition 5.1. *A linear connection $\mathcal{D}$ on TM is a d-connection if and only if it preserves by parallel transport the vertical subbundle.*

The proof is the same as in [5]. It follows from here

Proposition 5.2. *A connection on the manifold TM is a d-connection if and only if it satisfies one of the following conditions:*

$$\text{(40)} \qquad \text{(i) } p \circ \mathcal{D}_A \circ i = 0; \quad \text{(ii) } \sigma \circ \mathcal{D}_A \circ \sigma = 0, \quad \forall A \in \mathcal{T}^1(TM).$$

6. Derivation laws in the algebra of N-decomposable tensor fields.

Definition 6.1. A *vertical- horizontal* (vh)-connection on the manifold TM, with respect to a normalization N, is a linear connection on TM which induces a law of derivation in the vertical-horizontal subalgebra $VH\mathcal{T}(TM, N)$.

It is easy to prove

Proposition 6.1. *A connection $\mathcal{D}$ on TM is a vh-connection if and only if it satisfies one of the following conditions:*

$$\text{(41)} \qquad \begin{array}{ll} \text{(i)} & \mathcal{D} \text{ preserves the vertical subbundle,} \\ \text{(ii)} & \sigma \circ \mathcal{D}_A \circ \sigma = 0, \quad \text{(iii)} \ \ H \circ \mathcal{D}_A \circ V = 0, \ \forall A \in \mathcal{T}^1(TM). \end{array}$$

Definition 6.2. A *horizontal-vertical* (hv)-connection on TM, with respect to a normalization N, is a linear connection on TM, which induces a law of derivation in the horizontal–vertical subalgebra $HV\mathcal{T}(TM, N)$.

We obtain

Proposition 6.2. *A linear connection $\mathcal{D}$ on TM is a hv-connection if and only if it satisfies one of the following conditions:*

$$\begin{array}{ll} \text{(i)} & \mathcal{D} \text{ preserves the horizontal subbundle,} \\ \text{(ii)} & \tau \circ \mathcal{D}_A \circ \tau = 0, \quad \text{(iii)} \ \ V \circ \mathcal{D}_A \circ H = 0, \ \forall A \in \mathcal{T}^1(TM). \end{array}$$

Remark 6.1. A vh-connection is independent on the normalization and it is in the same time a d-connection. A hv-connection depends on the normalization N.

Let $\mathcal{D}$ be a linear connection on TM which preserves the vertical subbundle VTM and $A \in \mathcal{T}^1(TM)$, $B \in H\mathcal{T}^1(TM, N)$, $\alpha \in V\mathcal{T}_1(TM, N)$. We obtain $(\mathcal{D}_A\alpha)(B) = -\alpha(\mathcal{D}_A B)$ and so, $\mathcal{D}_A\alpha$ is a vertical 1-form if and only if $\mathcal{D}_A B$ is a horizontal vector field. Therefore, a connection $\mathcal{D}$ on TM induces a law of derivation in the subalgebra of vertical tensor fields (i.e. it is *vertical*) if and only if it induces a law of derivation in the subalgebra of horizontal tensor fields (i.e. it is *horizontal*). But in this case it induces a law of derivation in the whole algebra of N-decomposable tensor fields. Hence it is justified the following

Definition 6.3. A N-*decomposable* connection, with respect to a normalization N, is a linear connection $\mathcal{D}$ on TM, which induces a law of derivation in the fourgraded algebra of N-decomposable tensor fields $\mathcal{T}(TM, N)$.

From the previous considerations it follows

Proposition 6.3. *A linear connection $\mathcal{D}$ on the manifold TM is N-decomposable if and only if it satisfies one of the conditions:*

(i) *$\mathcal{D}$ is in the same time a vh and a hv-connection.*
(ii) *$\mathcal{D}$ induces a law of derivation in the vertical subalgebra.*
(iii) *$\mathcal{D}$ induces a law of derivation in the horizontal subalgebra.*
(iv) *$\mathcal{D}$ is a F-connection, i.e. $\mathcal{D}F = 0$.*
(v) *There exists a pair of connections $(\widetilde{\mathcal{D}}, \approx{\mathcal{D}})$ on the subbundles VTM and HTM, so that*

$$\mathcal{D}_A = \widetilde{\mathcal{D}}_A \circ V + \overset{\approx}{\mathcal{D}} \circ H, \ \forall A \in \mathcal{T}^1(TM). \tag{42}$$

Let be N a normalization on TM, P the second almost paracomplex structure associated to N and $\widetilde{\mathcal{D}}$ a linear connection on the vertical subbundle VTM. Then setting

$$\overset{\approx}{\mathcal{D}}_{X^h} Y^h = P(\widetilde{\mathcal{D}}_{X^h} Y^v), \ \overset{\approx}{\mathcal{D}}_{X^v} Y^h = 0, \quad \forall X, Y \in \mathcal{T}^1(M), \tag{43}$$

we obtain a linear connection on the horizontal subbundle HTM. Conversely, if $\overset{\approx}{\mathcal{D}}$ is a linear connection on HTM, then putting

$$\widetilde{\mathcal{D}}_{X^h} Y^v = P(\overset{\approx}{\mathcal{D}}_{X^h} Y^h), \ \widetilde{\mathcal{D}}_{X^v} Y^v = 0, \quad \forall X, Y \in \mathcal{T}^1(M), \tag{44}$$

we obtain a linear connection on VTM. So, we get

Proposition 6.4. *A N-decomposable connection on TM is uniquely determined by a linear connection on VTM or on HTM.*

It is easy to prove

Proposition 6.5. *If N is a normalization on the manifold TM, then setting*

$$\begin{aligned} &\widetilde{\mathcal{D}}_{X^h} Y^v = [X^h, Y^v], \ \widetilde{\mathcal{D}}_{X^v} Y^v = 0; \\ &\overset{\approx}{\mathcal{D}}_{X^h} Y^h = P[X^h, Y^v], \ \overset{\approx}{\mathcal{D}}_{X^v} Y^h = 0, \end{aligned} \tag{45}$$

$\forall X, Y \in \mathcal{T}^1(TM)$, one obtains a pair $(\widetilde{\mathcal{D}}, \overset{\approx}{\mathcal{D}})$ of linear connections on VTM and HTM and by (42) a N-decomposable connection on TM.

Definition 6.4. The N-decomposable connection $\mathcal{D}$ on TM given by

$$\mathcal{D}_{X^h} Y^v = [X^h, Y^v], \ \mathcal{D}_{X^v} Y^v = 0, \ \mathcal{D}_{X^h} Y^h = P[X^h, Y^v], \ \mathcal{D}_{X^v} Y^h = 0, \tag{46}$$

will be called the *canonical* connection associated to the normalization N.

For its local expression, we obtain

$$\mathcal{D}_{\delta_j} \dot{\partial}_k = \frac{\partial N_j^i}{\partial y^k} \dot{\partial}_i, \ \mathcal{D}_{\dot{\partial}_j} \dot{\partial}_k = 0, \ \mathcal{D}_{\delta_j} \delta_k = \frac{\partial N_j^i}{\partial y^k} \delta_i, \ \mathcal{D}_{\dot{\partial}_j} \delta_k = 0. \tag{47}$$

The torsion $\mathcal{T}$ and the curvature $\mathcal{R}$ of $\mathcal{D}$ are given by

$$\mathcal{T}(X^h, Y^h) = -N_P(X^h, Y^h), \ \mathcal{T}(X^h, Y^v) = 0, \ \mathcal{T}(X^v, Y^v) = 0, \tag{48}$$

where N_P is the Nijenhuis tensor field of P and

$$(49)\qquad \begin{array}{ll} \mathcal{R}_{X^hY^h}Z^v = [V[X^h, Y^h], Z^v], & \mathcal{R}_{X^hY^h}Z^h = P[V[X^h, Y^h], Z^v], \\ \mathcal{R}_{X^hY^v}Z^v = [[X^h, Z^v], Y^v], & \mathcal{R}_{X^hY^v}Z^h = P[[X^h, Z^v], Y^v], \\ \mathcal{R}_{X^vY^v}Z^v = 0, & \mathcal{R}_{X^vY^v}Z^h = 0. \end{array}$$

We obtain also the following useful relations

$$(50)\qquad \begin{array}{l} \mathcal{D}_{X^v}K = X^v, \quad \mathcal{D}_{X^h}K = [X^h, K], \\ \mathcal{D}V = \mathcal{D}H = \mathcal{D}F = \mathcal{D}P = \mathcal{D}J = \mathcal{D}\sigma = \mathcal{D}\tau = 0. \end{array}$$

It follows

Proposition 6.6. *If $\mathcal{D}$ is the canonical connection associated to normalization N on TM, then the canonical vector field K is absolute concurrent for vertical movements. It is absolute parallel for horizontal movements if and only if N is defined by a linear connection on M.*

A characterization for the canonical connection is given by

Proposition 6.7. *The linear connection $\mathcal{D}$ on the tangent manifold TM is the canonical connection, associated to the normalization N, if and only if it satisfies the conditions:*

$$(51)\qquad \mathcal{D}F = 0, \quad \mathcal{D}P = 0, \quad \mathcal{T} \circ V \times H = 0.$$

Proof. If $\mathcal{D}$ is the canonical connection, then the conditions are satisfied by (46) and (50). Conversely, from $\mathcal{D}F = 0$, it follows that $\mathcal{D}$ preserves the vertical and horizontal subbundles. After that, $\mathcal{T} \circ V \times H = 0$ gives $\mathcal{D}_{X^h}Y^v - \mathcal{D}_{Y^v}X^h - [X^h, Y^v] = 0$. But from $\mathcal{D}_{X^h}Y^v$, $[X^h, Y^v] \in VT^1(TM)$, and $\mathcal{D}_{Y^v}X^h \in HT^1(TM, N)$, it follows that $\mathcal{D}_{X^h}Y^v = [X^h, Y^v]$ and $\mathcal{D}_{Y^v}X^h = 0$. Combining these with $\mathcal{D}P = 0$, it follows $\mathcal{D}_{X^h}Y^h = P[X^h, Y^v]$ and $\mathcal{D}_{X^v}Y^h = 0$.

If we take, in particular, the normalization N, given by a linear connection ∇ on M, we may set the following

Definition 6.5. The *ν-lift* for a linear connection ∇ on M is the canonical connection on TM associated to the normalization N defined by ∇, that is, the connection $\mathcal{D} = \nabla^\nu$ given by

$$(52)\qquad \nabla^\nu_{X^h}Y^v = (\nabla_X Y)^v, \ \nabla^\nu_{X^v}Y^v = 0, \ \nabla^\nu_{X^h}Y^h = (\nabla_X Y)^h, \ \nabla^\nu_{X^v}Y^h = 0.$$

For the torsion and the curvature of ∇^ν, we obtain

$$(53)\qquad \begin{array}{l} \mathcal{T}(X^h, Y^h) = T^\nabla(X, Y)^h + \gamma R^\nabla_{XY}, \ \mathcal{T}(X^h, Y^v) = 0, \\ \mathcal{T}(X^v, Y^v) = 0, \ \mathcal{R}_{X^hY^h}Z^v = (R^\nabla_{XY}Z)^v, \\ \mathcal{R}_{X^hY^h}Z^h = (R^\nabla_{XY}Z)^h, \ \mathcal{R}_{X^hY^v} = 0, \ \mathcal{R}_{X^vY^v} = 0, \end{array}$$

where T^∇ and R^∇ are the torsion and the curvature of ∇.

The previous construction may be extended as follows.

Definition 6.6. The *ν_0-lift* of a linear connection ∇ on M, with respect to another linear connections ∇^0 on M, is the N_0-decomposable connection $\mathcal{D}^{\nu_0}$ on TM, given by

$$(54)\qquad \begin{array}{ll} \mathcal{D}^{\nu_0}_{X^{h_0}}Y^v = (\nabla_X Y)^v, & \mathcal{D}^{\nu_0}_{X^v}Y^v = 0, \\ \mathcal{D}^{\nu_0}_{X^{h_0}}Y^{h_0} = (\nabla_X Y)^{h_0}, & \mathcal{D}^{\nu_0}_{X^v}Y^{h_0} = 0, \end{array}$$

where h_0 is the horizontal lift corresponding to ∇^0.

For the torsion and the curvature of $\mathcal{D}^{\nu_0}$ we obtain

$$
\begin{aligned}
&\mathcal{T}(X^{h_0},Y^{h_0})=T^{\nabla}(X,Y)^{h_0}+\gamma R^{\nabla^0}_{XY},\\
&\mathcal{T}(X^{h_0},Y^{v}) = S(X,Y)^{v},\ \mathcal{T}(X^{v},Y^{v})=0,\\
&\mathcal{R}_{X^{h_0}Y^{h_0}}Z^{v}=(R^{\nabla}_{XY}Z)^{v},\ \mathcal{R}_{X^{h_0}Y^{h_0}}Z^{h_0}=(R^{\nabla}_{XY}Z)^{h_0},\\
&\mathcal{R}_{X^{h_0}Y^{v}}=0,\ \mathcal{R}_{X^{v}Y^{v}}=0,
\end{aligned}
\tag{55}
$$

where $S = \nabla - \nabla^0$.

Remark 6.1. The "horizontal lift" for a linear connection ∇ on M, defined by K. Yano and S. Ishihara [15], is the ν_0-lift $\mathcal{D}^{\nu_0}$ of ∇, with respect to the normalization N^0 defined by the transposed connection of ∇, i.e. the connection $\nabla^0=\nabla-T^{\nabla}$. In this case we have in (55), $S=T^{\nabla}$. The denomination "horizontal lift of ∇", given in [15] for $\mathcal{D}^{\nu_0}$ defined by $\nabla^0 = \nabla - T^{\nabla}$, is not justified because generally $\mathcal{D}^{\nu_0}$ does not induces a law of derivation in the horizontal subalgebra on TM with respect to the normalization N, defined by ∇. After that, $\mathcal{D}^{\nu_0}$ is more complicated than $\mathcal{D}^{\nu}$.

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