34 Almost hyperproduct structures on manifolds

Dedicated to Acad. Prof. Dr. Radu MIRON on the occasion of his 75th birthday

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1. Introduction. The almost hyperproduct (ahp)-structures on a manifold were considered, together with other important structures, by Liberman [6]. New properties of these structures were established by Walker [9, 10]. Legrand [5], Hsu [4], Vidal and Vidal Costa [8] and others, in the more general setting of the r-π-structures on manifolds.

In this paper, we give a new definition for an ahp-structure and we establish its equivalence with other geometric structures. We study then the compatibility of the ahp-structures with metrics and linear connections, their integrability and we determine two canonical connections compatible with a metric ahp-structure. Finally, we give as example, a metric ahp-structure on the tangent bundle, obtained by lifting a metric almost product structure off the base manifold.

2. Hyperproduct structures on a vector space

Definition 2.1. A *hyperproduct* (hp)-*structure* on a real vector space V is a triple (F, G, H) of automorphisms of V which satisfy the following conditions:

1) $F^2 = G^2 = H^2 = F \circ G \circ H = I$

2) I, F, G, H are linear independent.

It results from here

3) $F \circ G = G \circ F = H,\ G \circ H = H \circ G = F,\ H \circ F = F \circ H = G.$

Considering the projectors $F_1 = \dfrac{I+F}{2}$, $F_2 = \dfrac{I-F}{2}$, etc., we obtain

4) $F_a \circ G_b = G_b \circ F_a,\ G_a \circ H_b = H_b \circ G_a,\ H_a \circ F_b = F_b \circ H_a,\ a, b = 1, 2$

and

5) $F_1 \circ G_1 = G_1 \circ H_1 = H_1 \circ F_1,\ F_1 \circ G_2 = G_2 \circ H_2 = H_2 \circ F_1,$
$F_2 \circ G_1 = G_1 \circ H_2 = H_2 \circ F_2,\ F_2 \circ G_2 = G_2 \circ H_1 = H_1 \circ F_2.$

From these relations and $(F_1 + F_2) \circ (G_1 + G_2) = I$, it follows

$$F_1 \circ G_1 + F_1 \circ G_2 + F_2 \circ G_1 + F_2 \circ G_2 = I$$

and setting

6) $P_1 = F_1 \circ G_1,\ P_2 = F_1 \circ G_2,\ P_3 = F_2 \circ G_1,\ P_4 = F_2 \circ G_2,$

one obtains

7) $\sum_{\alpha=1}^{4} P_\alpha = I,\ P_\alpha^2 = P_\alpha,\ P_\alpha \circ P_\beta = 0,\ \alpha \neq \beta = 1,2,3,4.$

Therefore, P_α are independent and supplementary projectors on V. Setting then

8) $W_\alpha = P_\alpha(V),\ \alpha = 1,2,3,4,$

it results that W_α are independent and supplementary subspaces of V, i.e.

9) $V = W_1 \oplus W_2 \oplus W_3 \oplus W_4.$

Denoting with (F^+, F^-), $G^+, G^-)$, (H^+, H^-) the eigensubspaces of F, G, H corresponding to $+1$ and -1, it follows

10) $$\begin{aligned} W_1 &= F^+ \cap G^+ = G^+ \cap H^+ = H^+ \cap F^+, \\ W_2 &= F^+ \cap G^- = G^- \cap H^- = H^- \cap F^+ \\ W_3 &= F^- \cap G^+ = G^+ \cap H^- = H^- \cap F^-, \\ W_4 &= F^- \cap G^- = G^- \cap H^+ = H^+ \cap F^-. \end{aligned}$$

From here it results

11) $$\begin{aligned} &F^+ = W_1 \oplus W_2,\ F^- = W_3 \oplus W_4,\ G^+ = W_1 \oplus W_3, \\ &G^- = W_2 \oplus W_4,\ H^+ = W_1 \oplus W_4,\ H^- = W_2 \oplus W_3. \end{aligned}$$

Setting $n_\alpha = \dim W_\alpha$, we obtain $n_1 + n_2 = \dim F^+$, $n_3 + n_4 = \dim F^-$, etc., and $n_1 + n_2 + n_3 + n_4 = \dim V$. If F, G, H are paracomplex (pc)-structures, i.e. $\operatorname{tr} F = \operatorname{tr} G = \operatorname{tr} H = 0$, then $n_1 = n_2 = n_3 = n_4 = n$ and $\dim V = 4n$. We remark that if two of the structures F, G, H are paracomplex, the rested is not necessary paracomplex.

Let $Q_\alpha = I - P_\alpha$ be the supplementary projector of P_α and $\phi_\alpha = 2P_\alpha - I$, $\alpha = 1,2,3,4$, be the associated product structures. Denoting $\overline{W}_\alpha = \sum_{\beta \neq \alpha} W_\alpha$, we obtain for the eigensubspaces of ϕ_α

12) $\phi_\alpha^+ = W_\alpha,\ \phi_\alpha^- = \overline{W}_\alpha,\ \alpha = 1,2,3,4.$

We have also the relations

13) $\phi_\alpha^2 = I,\ \phi_\alpha \circ \phi_\beta = \phi_\beta \circ \phi_\alpha,\ \alpha \neq \beta = 1,2,3,4,\ \phi_1 \circ \phi_2 \circ \phi_3 \circ \phi_4 = -I,$

14) $F = -\phi_1 \circ \phi_2 = \phi_3 \circ \phi_4,\ G = -\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4,\ H = -\phi_1 \circ \phi_4 = \phi_2 \circ \phi_3.$

We remark the useful relations

15) $$\begin{aligned} &4P_1 = I + F + G + H,\ 4P_2 = I + F - G - H, \\ &4P_3 = I - F + G - H,\ 4P_4 = I - F - G + H \end{aligned}$$

and

16) $F = P_1 + P_2 - P_3 - P_4,\ G = P_1 - P_2 + P_3 - P_4,\ G = P_1 - P_2 - P_3 + P_4.$

Considering on F^+ and F^- the product structures φ and ψ given by $\varphi^+ = W_1$, $\varphi^- = W_2$, $\psi^+ = W_3$, $\psi^- = W_4$, we obtain for G and H

17) $G = \varphi \circ F_1 + \psi \circ F_2,\ H = \varphi \circ F_1 - \psi \circ F_2.$

From the previous considerations it follows that to an *hp*-structure on V we can associate the following systems of subspaces $\{W_\alpha\}$, $\{W_{\alpha\beta} = W_\alpha \oplus W_\beta\}$, $\{\overline{W}_\alpha\}$, $\alpha \neq \beta = 1,2,3,4$. We can associate also some equivalent structures: $\{P_\alpha\}$, $\{\phi_\alpha\}$, $\alpha = 1,2,3,4$, $\{F,\varphi,\psi\}$, which must satisfy certain conditions resulting from 1) and 2).

In a *canonical* basis for the *hp*-structure (F,G,H), that is formed by vectors situated in W_α, $\alpha = 1,2,3,4$, we obtain for F, G, H the matrices

$$F = \begin{bmatrix} I & & & \\ & I & & \\ & & -I & \\ & & & -I \end{bmatrix},\quad G = \begin{bmatrix} I & & & \\ & -I & & \\ & & I & \\ & & & -I \end{bmatrix},\quad H = \begin{bmatrix} I & & & \\ & -I & & \\ & & -I & \\ & & & I \end{bmatrix},$$

with the diagonal blocs formed by the $(\pm)$-unitary matrices of dimensions n_α, $\alpha = 1,2,3,4$. It follows from here that the group of automorphisms for the *hp*-structure (F,G,H) is isomorphic with $GL(n_1,\mathbb{R}) \times GL(n_2,\mathbb{R}) \times GL(n_3,\mathbb{R}) \times GL(n_4,\mathbb{R})$.

For a metric h on V, i.e. a symmetric and nondegenerate $(0,2)$-tensor, we consider the metrics

19) $g_1 = h \circ (I \times I + F \times F + G \times G + H \times H),\ g_2 = g_1 \circ (F \times I),$
$g_3 = g_1 \circ (G \times I),\ g_4 = g_1 \circ (H \times I).$

It follows

20) $g_\alpha \circ (F \times F) = g_\alpha \circ (G \times G) = g_\alpha \circ (H \times H) = g_\alpha,\ \alpha = 1,2,3,4.$

Let $\mathcal{M} = \{g_1, g_2, g_3, g_4\}$ be the set of previous metrics and $\mathcal{G} = \{I, F, G, H\}$ the subgroup of $GL(V)$ determined by the *hp*-structure (F,G,H). Setting

$$T(g,(u,v)) = g \circ (u \times v),\ \forall g \in \mathcal{M},\ u, v \in \mathcal{G},$$

one obtains a right action of the group $\mathcal{G} \times \mathcal{G}$ on $\mathcal{M}$. The elements of $\mathcal{M}$ are invariant to the restriction of the action T of $\mathcal{G} \times \mathcal{G}$ to its diagonal subgroup.

Definition 2.2. We call the structure (F,G,H,g_1) a *metric hyperproduct* (*mhp*)-structure on V and g_2, g_3, g_4 the *associated* metrics.

If h is an Euclidean metric, then g_1 is also Euclidean and g_2, g_3, g_4 are pseudo–Euclidean. If F, G, H are *pc*-structures and h is Euclidean metric, then the structures (F,g_1), (G,g_1), (H,g_1) are Euclidean *mpc*-structures. Setting $\overset{\alpha}{g} = g_1/W_\alpha$, $\alpha = 1,2,3,4$, we obtain a metric on each of the subspaces W_α. All $\overset{\alpha}{g}$ are Euclidean when g_1 is Euclidean. The pairs (F^+,F^-), (G^+,G^-), (H^+,H^-) are then formed by orthogonal subspaces of V, with respect to all g_α and W_α are also orthogonal subspaces for all g_α. If g_1 is an Euclidean metric on V, then in an orthogonal basis on V, formed by vectors situated in W_α, we obtain

$$21)\ g_1 = \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix},\ g_2 = \begin{bmatrix} I & & & \\ & I & & \\ & & -I & \\ & & & -I \end{bmatrix},\ g_3 = \begin{bmatrix} I & & & \\ & -I & & \\ & & I & \\ & & & -I \end{bmatrix},\ g_4 = \begin{bmatrix} I & & & \\ & -I & & \\ & & -I & \\ & & & I \end{bmatrix}.$$

Hence, for an Euclidean *mhp*-structure (F, G, H, g_1), the automorphism group is isomorphic with $\mathcal{O}(n_1, \mathbb{R}) \times \mathcal{O}(n_2, \mathbb{R}) \times \mathcal{O}(n_3, \mathbb{R}) \times \mathcal{O}(n_4, \mathbb{R})$.

Almost hyperproduct structures on a manifold. Let M be a paracompact C^∞–manifold, $\mathcal{F}(M)$ the ring of real functions, $\mathcal{D}^r_s(M)$ the $\mathcal{F}(M)$ module of (r, s) tensor fields, $\mathcal{D}(M)$ the $\mathcal{F}(M)$-tensor algebra and $\mathcal{D}er(M)$ the $\mathcal{F}(M)$-module of derivations for $\mathcal{D}(M)$.

Definition 3.1. An almost hyperproduct (*ahp*)-structure on M is a triple (F, G, H) of (1,1) tensor fields, which satisfies the conditions 1) and 2) for each $x \in M$.

All the considerations from the previous section may be extended to the tangent bundle TM. For an *ahp*-structure (F, G, H) on M, we denote by $F^\pm, G^\pm, H^\pm$ the eigendistributions (or subbundles of TM), corresponding to ± 1 and by F_a, G_a, H_a, $a = 1, 2$, the projectors of F, G, H on $F^\pm$, $G^\pm$, $H^\pm$ respectively. We consider also the projectors P_α, $\alpha = 1, 2, 3, 4$, given by 6), the supplementary projectors $Q_\alpha = I - P_\alpha$ and the distributions (subbundles of TM)

22) $W_\alpha = P_\alpha(TM),\ W_{\alpha\beta} = W_\alpha \oplus W_\beta,\ \overline{W}_\alpha = \sum_{\beta\neq\alpha} W_\beta,\ \alpha \neq \beta = 1, 2, 3, 4.$

These distributions are related to the eigendistibutions of F, G, H by the relations 10) and 11) and they will be called the *distributions* of the *ahp*-structure (F, G, H). We denote by $\mathcal{D}^1(M, W_\alpha)$ the $\mathcal{F}(M)$-module of the sections of the subbundle W_α.

If ∇° is a connection on M, then each connection ∇ on M may be written in the form

23) $\nabla = \nabla^\circ + \tau,$

where τ is an $(1, 2)$ tensor field on M. That is, for $X \in \mathcal{D}^1(M)$ we have

$$\nabla_X = \nabla^\circ_X + \tau_X,$$

where ∇_X and ∇°_X are derivations in $\mathcal{D}(M)$ and τ_X is the $(1, 1)$ tensor field given by $\tau_X(Y) = \tau(X, Y)$, or τ_X is a derivation in $\mathcal{D}(M)$ with $\tau_X(f) = 0$ for each $f \in \mathcal{F}(M)$. From here it follows the useful result.

Proposition 3.1. *The set $\mathcal{C}(M)$ of the connections on M is an $\mathcal{F}(N)$-affine module (space)* [1] *associated to $\mathcal{F}(M)$-linear module $\mathcal{D}^1_2(M)$.*

Let F be an almost product (*ap*)-structure on M. Setting

24) $\psi_F(\nabla)_X = \dfrac{1}{2}(\nabla_X + F \circ \nabla_X \circ F),\ \chi_F(\tau) = \dfrac{1}{2}(\tau_X + F \circ \tau_X \circ F),$

$\forall X \in \mathcal{D}^1(M)$, it follows $\psi_F(\nabla) \in \mathcal{C}(M)$, $\chi_F(\tau) \in \mathcal{D}^1_2(M)$ and

25) $\psi_F^2 = \psi_F,\ \chi_F^2 = \chi_F,\ \psi_F(\nabla + \tau) = \psi_F(\nabla) + \chi_F(\tau).$

Hence, ψ_F is the $\mathcal{F}(M)$-affine projector on $\mathcal{C}(M)$, associated to $\mathcal{F}(M)$-linear projector χ_F on $\mathcal{D}^1_2(M)$.

Definition 3.2. A connection ∇ on M is called *compatible* with the *ap*-structure F (or is an *F-connection*) if it satisfies

26) $\nabla F = 0.$

It is easy to see that $\nabla F = 0$ if and only if ∇ preserves by parallelism the eigendistributions F^+ and F^- of F. From the expression of $\psi_F(\nabla)$ it results

27) $\psi_F(\nabla)_X(F) = 0, \ \forall X \in \mathcal{D}^1(M)$,

i.e. the image of any connection ∇ by the projector ψ_F is an F-connection. Conversely, if $\nabla_X F = 0$ it follows $\nabla_X \circ F - F \circ \nabla_X = 0$ and so, $\psi_F(\nabla)_X = \nabla_X, \ \forall X \in \mathcal{D}^1(M)$ i.e. $\nabla \in \operatorname{Im} \psi_F$. Thus, we have

Theorem 3.1. *The set $\mathcal{C}_F(M)$ of the connections compatible with the ap-structure F is the affine submodule of $\mathcal{C}(M)$, which is the image of the affine projector ψ_F*

28) $\mathcal{C}_F(M) = \operatorname{Im} \psi_F$.

Considering on $\mathcal{C}(M)$ the *conjugation* with respect to F i.e. the $\mathcal{F}(M)$-automorphism $C_F : \mathcal{C}(M) \to \mathcal{C}(M)$ given by

29) $C_F(\nabla)_X = F \circ \nabla_X \circ F, \ \forall X \in \mathcal{D}^1(M)$,

we obtain

30) $\psi_F(\nabla) = \dfrac{1}{2}(\nabla + C_F(\nabla))$,

i.e. C_F is the affine symmetry of the affine module $\mathcal{C}(M)$ with respect to affine submodule $\mathcal{C}_F(M)$, made parallely with the linear submodule $\operatorname{Ker} \chi_F$. Hence, $\psi_F(\nabla)$ is the *mean* connection of ∇ and its conjugate with respect to F. $\psi_F(\nabla)$ will be called the *F-connection associated to* ∇ with respect to *ap*-structure F.

Let ∇° be a fixed connection on M. Since $\mathcal{C}_F(M) = \operatorname{Im} \psi_F$, then for each connection $\nabla \in \mathcal{C}_F(M)$ there exists $\nabla' \in \mathcal{C}(M)$ so that $\nabla = \psi_F(\nabla')$. But from 23), there exists $\tau \in \mathcal{D}^1_2(M)$ so that $\nabla' = \nabla^\circ + \tau$ and therefore, $\nabla = \psi_F(\nabla^\circ + \tau)$. Then from 25) it follows

Theorem 3.2. *The set $\mathcal{C}_F(M)$ of the connections ∇ compatible with the ap-structure F is given by*

31) $\nabla = \psi_F(\nabla^\circ) + \chi_F(\tau)$,

where ∇° is a fixed connection on M and τ is an arbitrary $(1,2)$-tensor field on M.

Hence $\mathcal{C}_F(M)$ is the affine submodule of $\mathcal{C}(M)$ which passes through the F-connection $\psi_F(\nabla^\circ)$ and has as direction the linear submodule $\operatorname{Im} \chi_F$ of $\mathcal{D}^1_2(M)$.

Definition 3.3. A connection ∇ is called *compatible* with the *ahp*-structure (F, G, H) (or is an *(F, G, H)-connection*) if it satisfies

32) $\nabla F = 0, \ \nabla G = 0, \ \nabla H = 0$.

It is clear that if we have, for example, $\nabla F = 0, \ \nabla G = 0$, then we have also

33) $\nabla H = 0, \ \nabla F_a = 0, \ \nabla G_a = 0, \ \nabla H_a = 0, \ a = 1, 2,$
$\nabla P_\alpha = 0, \ \nabla Q_\alpha = 0, \ \alpha = 1, 2, 3, 4.$

The compatibility of ∇ with the *ahp*-structure (F, G, H) is equivalent with the absolute parallelism of the eigendistributions of F, G and H, in the connection ∇. But it is easy to show that all the distributions $W_\alpha, W_{\alpha\beta}, \overline{W}_\alpha$, $\alpha \neq \beta = 1, 2, 3, 4$ are absolute parallel in the connection ∇ if and only if W_α, $\alpha = 1, 2, 3, 4$ are. Hence we have

Theorem 3.3. *The connection ∇ is compatible with the ahp-structure (F, G, H) if and only if it preserves by parallelism the distributions W_α, $\alpha = 1, 2, 3, 4$.*

From $F \circ G = G \circ F$ it follows $\psi_F \circ \psi_G = \psi_G \circ \psi_F$, $\chi_F \circ \chi_G = \chi_G \circ \chi_F$, $C_F \circ C_G = C_G \circ C_F$. After that ψ_F and ψ_G being affine projectors associated to linear projectors χ_F and χ_G, it follows that $\psi_F \circ \psi_G$ is the affine projector associated to linear projector $\chi_F \circ \chi_G$, i.e.,

34) $\psi_F \circ \psi_G(\nabla + \tau) = \psi_F \circ \psi_G(\nabla) + \chi_F \circ \chi_G(\tau)$.

From here it results

Theorem 3.4. *The set $\mathcal{C}_{F,G}(M)$ of the connections compatible with the ap-structure (F, G, H) is given by*

35) $\nabla = \psi_F \circ \psi_G(\nabla^\circ) + \chi_F \circ \chi_G(\tau)$,

where ∇° is a fixed connection on M and τ is an arbitrary element of $\mathcal{D}^1_2(M)$.

With other words, the set $\mathcal{C}_{F,G}(M)$, of (F, G, H)-connections on M, is the image of the affine projector $\psi_F \circ \psi_G$

36) $\mathcal{C}_{F,G}(M) = \mathrm{Im}(\psi_F \circ \psi_G)$,

i.e. it is the affine submodule of $\mathcal{C}(M)$ passing by the (F, G, H)-connection $\psi_F \circ \psi_G(\nabla^\circ)$, which has the direction given by the linear submodule $\mathrm{Im}(\chi_F \circ \chi_G)$ of $\mathcal{D}^1_2(M)$.

Taking in 35) $\tau = 0$, it follows that an *ahp*-structure (F, G, H) assigns to each connection $\nabla^\circ \in \mathcal{C}(M)$, an (F, G, H)-connection $\nabla = \psi_F \circ \psi_G(\nabla^\circ)$, which may be written in the form

37) $\nabla = \frac{1}{4}[\nabla^\circ + C_F(\nabla^\circ) + C_G(\nabla^\circ) + C_H(\nabla^\circ)]$.

Hence, we have

Proposition 3.2. *The (F, G, H)-connection ∇ associated to a connection ∇° is the mean connection of ∇° and its conjugate connections with respect to ap-structures F, G, H.*

If we consider $X \in \mathcal{D}^1(M)$ and $Y_\alpha \in \mathcal{D}^1(M, W_\alpha)$, $\alpha = 1, 2, 3, 4$, we obtain from 37), taking into account 15),

38) $\nabla_X Y_\alpha = P_\alpha(\nabla^\circ_X Y_\alpha)$, $\alpha = 1, 2, 3, 4$.

Hence, we have

Proposition 3.3. *The connections ∇^α, $\alpha = 1, 2, 3, 4$, induced by the (F, G, H)-connection ∇, associated to ∇°, on the subbundles W_α by restriction, coincide with the projections of ∇° on W_α.*

Taking $Y \in \mathcal{D}^1(M)$ and setting $Y = \sum_{\alpha=1}^{4} Y_\alpha$, with $Y_\alpha \in \mathcal{D}^1(M, W_\alpha)$, we get

39) $\nabla_X Y = \sum_{\alpha=1}^{4} P_\alpha(\nabla^\circ_X(P_\alpha Y)))$

i.e.

40) $\nabla_X = \sum_{\alpha=1}^4 P_\alpha \circ \nabla^\circ_X \circ P_\alpha,\ \forall X \in \mathcal{D}^1(M)$.

Definition 3.4. An *Otsuki quasiconnection* [7] on M is a pair $\mathcal{D} = (P, D)$, formed by a tensor field $P \in \mathcal{D}^1_1(M)$ and a mapping $D : \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M)$, which is $\mathcal{F}(M)$-linear in the first argument, $\mathbb{R}$-linear in the second and satisfies

$$D_X(fY) = X(f)P(Y) + fD_XY,\ \forall f \in \mathcal{F}(M),\ X, Y \in \mathcal{D}^1(M).$$

It results that a linear connection is an Otsuki quasiconnection with $P = I$. From 7) and (40) it follows

Proposition 3.4. *For a connection ∇° on M, the pairs $\mathcal{D}^\alpha_X = (P_\alpha, P_\alpha \circ \nabla^\circ_X \circ P_\alpha)$, $\alpha = 1,2,3,4$, $X \in \mathcal{D}^1(M)$ determine four Otsuki quasiconnections on M. The restrictions of $\mathcal{D}^\alpha$ to the subbundles W_α coincide with the connections ∇^α, obtained from ∇° by projections and the sum of $\mathcal{D}^\alpha$ is the (F,G,H)-connection ∇ associated to ∇°.*

From 38) and 40) it follows

Proposition 3.5. *A connection ∇ on M is an (F,G,H)-connection if and only if there are the connections ∇^α on the subbundles W_α so that*

41) $\nabla_X = \sum_{\alpha=1}^{4} \nabla^\alpha_X \circ P_\alpha,\ \forall X \in \mathcal{D}^1(M)$.

It results also from 40)

Proposition 3.6. *A vector field $Y \in \mathcal{D}^1_2(M)$ is parallel along a curve $\gamma \subset M$ in the (F,G,H)-connection ∇ associated to ∇°, if and only if its components $Y_\alpha \in \mathcal{D}^1(M, W_\alpha)$ are parallel along γ in the induced connections ∇^α on W_α.*

The problem of the integrability for an *ahp*-structure has been analysed in a more general framework by Walker [9,10], Hsu [4], Vidal and Vidal Costa [8] and others. We give here some characterizations specific for our case.

Definition 3.5. One sais that the *ahp*-structure (F,G,H) is *integrable* if all the distributions of the structure, i.e. $W_\alpha, W_{\alpha\beta}, \overline{W}_\alpha$, $\alpha = 1,2,3,4$, are integrable.

One has

Proposition 3.7. *The ahp-structure (F,G,H) is integrable if and only if the distributions $W_{\alpha\beta}$, $\alpha \neq \beta = 1,2,3,4$, are integrable.*

Indeed, from $W_\alpha = W_{\alpha\beta} \cap W_{\alpha\gamma}$, $\alpha \neq \beta \neq \gamma$ and $W_{\alpha\beta}$ integrable, it follows that W_α are integrable. After that, from $\overline{W}_\alpha = \sum_{\beta\neq\alpha} W_\beta$ and W_α, $W_{\alpha\beta}$ integrable, it results $\overline{W}_\alpha$ integrable.

Since $W_{\alpha\beta}$ are the eigendistributions of the *ap*-structures F, G and H, from this proposition and the integrability of an *ap*-structure it follows

Theorem 3.4. *The ahp-structure (F,G,H) is integrable if and only if the ap-structures F, G and H are integrable, i.e. their Nijenhuis tensors are zero*

42) $N_F = 0,\ N_G = 0,\ N_H = 0.$

It is easy to verify that this condition is equivalent with

43) $N_F = 0,\ N_G = 0,\ N_{F,G} = 0.$

Considering a connection ∇° and its associated connection ∇ with respect to *ahp*-structure (F, G, H), given by 37), we obtain for the torsion of ∇

$$T(X_\alpha, Y_\beta) = P_\beta(\nabla^\circ_{X_\alpha} Y_\beta) - P_\alpha(\nabla^\circ_{Y_\beta} X_\alpha) - [X_\alpha, Y_\beta].$$

From here it follows

$$P_\gamma(T(X_\alpha, Y_\beta)) = -P_\gamma[X_\alpha, Y_\beta],\ \forall \alpha \neq \beta \neq \gamma = 1, 2, 3, 4.$$

Remarking that $P_\gamma[X_\alpha, Y_\beta] = 0$, for $\gamma \neq \alpha \neq \beta$, is equivalent with the integrability of the distribution $W_{\alpha\beta}$, it results

Proposition 3.8. *The ahp-structure (F, G, H) is integrable if and only if there exists a connection ∇° on M so that the torsion of the associated connection ∇, with respect to structure (F, G, H), satisfies*

44) $P_\gamma \circ T \circ (P_\alpha \times P_\beta) = 0,\ \forall \alpha \neq \beta \neq \gamma = 1, 2, 3, 4.$

Let now g be a metric on M i.e. a symmetric and nondegenerate $(0, 2)$-tensor field.

Definition 3.6. One sais that the metric g is *invariant* to the *ahp*-structure (F, G, H), or that (F, G, H, g) is a *mahp–structure* on M if

45) $g \circ (F \times F) = g,\ g \circ (G \times G) = g,\ g \circ (H \times H) = g.$

As in the first section we can prove that on any paracompact manifold M there exist Riemannian metrics invariant to a given *ahp*-structure.

Setting

46) $g_2 = g \circ (F \times I),\ g_3 = g \circ (G \times I),\ g_4 = g \circ (H \times I)$

we obtain new metrics on M, invariant to *ahp*-structure (F, G, H), called the *associated metrics* to *mahp*-structure (F, G, H, g).

Definition 3.7. A connection ∇ on M is called *compatible* with the metric g if it satisfies

47) $\nabla g = 0.$

Considering g as a mapping from $\mathcal{D}^1(M)$ to $\mathcal{D}_1(M)$, which assigns to a vector field Y the 1-form $\omega = g(Y, \cdot)$, i.e. $\omega(Z) = g(Y, Z)$, for each $Z \in \mathcal{D}^1(M)$, we can associate to a connection $\nabla \in \mathcal{C}(M)$ and a tensor field $\tau \in \mathcal{D}^1_2(M)$, the affine and the linear projectors ψ_g and χ_g given respectively by

48) $\psi_g(\nabla)_X = \dfrac{1}{2}(\nabla_X + g^{-1} \circ \nabla_X \circ g),\ \chi_g(\tau)_X = \dfrac{1}{2}(\tau_X + g^{-1} \circ \tau_X \circ g).$

As for an *ap*-structure F, for a metric g, we obtain

Theorem 3.5. *The set $\mathcal{C}_g(M)$ of connections ∇ compatible with a metric g are given by*

49) $\nabla = \psi_g(\nabla^\circ) + \chi_g(\tau)$,

where $\nabla^\circ \in \mathcal{C}(M)$ *is fixed and* $\tau \in \mathcal{D}^1_2(M)$ *is arbitrary.*

If (F, G, H, g) is an *mahp*-structure on M, then $\psi_F \circ \psi_g = \psi_g \circ \psi_F$, etc., and $\chi_F \circ \chi_g = \chi_g \circ \chi_F$, etc. Thus from [1] it follows

Theorem 3.6. *The set of connections on* M*, compatible with the mahp-structure* (F, G, H, g) *is given by*

50) $\nabla = \psi_F \circ \psi_G \circ \psi_g(\nabla^\circ) + \chi_F \circ \chi_G \circ \chi_g(\tau)$,

where $\nabla^\circ \in \mathcal{C}(M)$ *is fixed and* $\tau \in \mathcal{D}^1_2(M)$ *is arbitrary.*

We remark that if a connection is compatible with the *mahp*-structure (F, G, H, g) it is also compatible with the associated metrics g_2, g_3, g_4. Taking in 50) $\nabla^\circ = \nabla^g$, the Levi-Civita connection of g, we have $\psi_g(\nabla^g) = \nabla^g$ and setting $\tau = 0$, we obtain

Proposition 3.9. *The connection* $\widetilde{\nabla} = \psi_F \circ \psi_G(\nabla^g)$*, i.e. the* (F, G, H, g)*-connection associated to Levi-Civita connection* ∇^g *of* g*, is compatible with the mahp-structure* (F, G, H, g) *on* M.

Definition 3.8. We call the connection $\widetilde{\nabla} = \psi_F \circ \psi_G(\nabla^g)$, the *first canonical* connection associated to *mahp*-structure (F, G, H, g).

Considering an (F, G, H)-connection ∇ on M, we have $\psi_F(\nabla) = \psi_G(\nabla)$ $= \nabla$ and therefore for the connection $\widetilde{\nabla} = \psi_F \circ \psi_G \circ \psi_g(\nabla)$, compatible with the *mahp*-structure (F, G, H, g), we obtain $\widetilde{\nabla} = \psi_g(\nabla)$. Hence, we have

Proposition 3.10. *If* ∇ *is an* (F, G, H)*-connection on* M*, then* $\overline{\nabla} = \psi_g(\nabla)$ *is compatible with the mahp-structure* (F, G, H, g).

Let ∇ be an (F, G, H)-connection on M. As we have seen, setting $\nabla^\alpha_X Y_\alpha = \nabla_X Y_\alpha$, for $X \in \mathcal{D}^1(M)$ and $Y_\alpha \in \mathcal{D}^1(M, W_{\alpha\beta})$, $\alpha = 1, 2, 3, 4$, we obtain a connection ∇^α on each subbundle W_α of TM. Considering then as torsion for the connection ∇^α, the tensor field $T^\alpha = P_\alpha \circ T \circ (P_\alpha \times P_\alpha)$ restricted to W_α, where T is the torsion of ∇, we obtain

51) $T^\alpha(X_\alpha, Y_\alpha) = \nabla_{X_\alpha} Y_\alpha - \nabla_{Y_\alpha} X_\alpha - P_\alpha[X_\alpha, Y_\alpha],\ \forall X_\alpha, Y_\alpha \in \mathcal{D}^1(M, W\alpha),\ \alpha = 1, 2, 3, 4.$

Now we can prove

Theorem 3.7. *Given a Riemannian mahp-structure* (F, G, H, g) *on* M*, there exists a unique connection* $\widehat{\nabla}$ *on* M *which satisfies*

52) a) $\widehat{\nabla} F = 0,\ \widehat{\nabla} G = 0$; b) $P_\beta \circ \widehat{T} \circ (P_\alpha \times P_\beta) = 0,\ \alpha \neq \beta$; c) $\widehat{T}^\alpha = 0$;
d) $\widehat{\nabla}_{X_\alpha} \overset{\alpha}{g} = 0$, *where* $\overset{\alpha}{g} = g \circ (P_\alpha \times P_\alpha)$ *and* $\alpha, \beta = 1, 2, 3, 4$.

Indeed, from a) we obtain $\widehat{\nabla}_X Y_\alpha \in \mathcal{D}^1(M, W_\alpha)$, $\forall X \in \mathcal{D}^1(M)$, $Y_\alpha \in \mathcal{D}^1(M, W_\alpha)$, $\alpha = 1, 2, 3, 4$, and taking into account b) it follows

53) $\widehat{\nabla}_{X_\alpha} Y_\beta = P_\beta[X_\alpha, Y_\beta],\ \alpha \neq \beta = 1, 2, 3, 4.$

After that, from c) and d) we have

54) $\widehat{\nabla}_{X_\alpha} Y_\alpha - \widehat{\nabla}_{Y_\alpha} X_\alpha - P_\alpha[X_\alpha, Y_\alpha] = 0,$
$X_\alpha \overset{\alpha}{g}(Y_\alpha, Z_\alpha) = \overset{\alpha}{g}(\widehat{\nabla}_{X_\alpha} Y_\alpha, Z_\alpha) + \overset{\alpha}{g}(X_\alpha, \widehat{\nabla}_{X_\alpha} Z_\alpha).$

Thus, as well as in the Riemannian case, it follows from here

$$2\overset{\alpha}{g}(\widehat{\nabla}_{X_\alpha}Y_\alpha, Z_\alpha) = X_\alpha\overset{\alpha}{g}(Y_\alpha, Z_\alpha) + Y_\alpha\overset{\alpha}{g}(Z_\alpha, X_\alpha) - Z_\alpha\overset{\alpha}{g}(Y_\alpha, Y_\alpha) -$$
$$-\overset{\alpha}{g}(X_\alpha, P_\alpha[Y_\alpha, Z_\alpha]) + \overset{\alpha}{g}(Y_\alpha, P_\alpha[Z_\alpha, X_\alpha]) + \overset{\alpha}{g}(Z_\alpha, P_\alpha[X_\alpha, Y_\alpha]),$$
$$\forall X_\alpha, Y_\alpha, Z_\alpha \in \mathcal{D}^1(M, W_\alpha).$$

Hence, we have obtained a unique connection $\widehat{\nabla}$ which preserves the *ahp*-structure (F, G, H), but generally it does not preserves the metric g. More precisely, we have

56) $(\widehat{\nabla}_{X_\alpha}g)(Y_\alpha, Z_\alpha) = 0,\ \widehat{\nabla}_{X_\alpha}g(Y_\beta, Z_\beta) = (\mathcal{L}_{X_\alpha}\overset{\alpha}{g})(Y_\beta, Z_\beta),$
$(\widehat{\nabla}_{X_\alpha}g)(Y_\beta, Z_\gamma) = 0,\ \alpha \neq \beta \neq \gamma = 1, 2, 3, 4,$

where $\mathcal{L}_X$ is the Lie derivative with respect to X.

Setting then $\overline{\nabla} = \psi_g(\widehat{\nabla})$, it follows from Proposition 3.10 that $\overline{\nabla}$ is an (F, G, H, g)-connection on M.

Definition 3.9. We call the connection $\overline{\nabla} = \psi_g(\widehat{\nabla})$ the *second canonical connection* for the Riemannian *mahp*-structure (F, G, H, g).

Example. Let N be a manifold and $M = TN$ the total space of the tangent bundle $\pi : TN \to N$. Setting for each 1-form $\mu \in \mathcal{D}_1(N)$, given locally by $\mu(x) = \mu_i(x)dx^i$, $\gamma(\mu)(z) = \mu_i(x)y^i$, where $z = (x, y) \in T_xN$, we obtain a class of functions on TN, with the following property. For any two vector fields $A, B \in \mathcal{D}^1(TN)$, we have $A = B$ if and only if $A(\gamma\mu) = B(\gamma\mu)$, for each $\mu \in \mathcal{D}_1(N)$.

Let ∇ be a connection and X a vector field on N. Setting

57) $X^h(\gamma\mu) = \gamma(\nabla_X\mu),\ X^v(\gamma\mu) = \mu(X) \circ \pi,\ \forall \mu \in \mathcal{D}_1(N),$

we obtain two vector fields X^h and X^v on TN, called respectively the *horizontal* and the *vertical lifts* of X. For an 1-form ω on N, the *horizontal* and *vertical lifts* are given by

58) $\omega^h(X^h) = \omega(X) \circ \pi,\ \omega^h(X^v) = 0;$
$\omega^v(X^h(= 0,\ \omega^v(X^v) = \omega(X) \circ \pi,\ \forall X \in \mathcal{D}^1(N).$

After that, setting

59) $F(X^h) = X^h,\ F(X^v) = -X^v,\ \forall X \in \mathcal{D}^1(N),$

we obtain an *apc*-structure F on TN, having as eigendistributions F^+ and F^-, the *horizontal distribution* HTN of the connection ∇ and the *vertical distribution* VTN of the fibration. For $f \in \mathcal{D}^1_1(TN)$ and $g \in \mathcal{D}^0_2(N)$ we define the *horizontal* and *vertical lifts* f^h, f^v and g^h, g^v on TN, by

60) $f^h(X^h) = f(X)^h,\ f^h(X^v) = 0;\ f^v(X^h) = 0,\ f^v(X^v) = f(X)^v,$
$g^h(X^h, Y^h) = g(X, Y) \circ \pi,\ g^h(X^h, Y^v) = g^h(X^v, Y^h) = g^h(X^v, Y^v) = 0,$
$g^v(X^h, Y^h) = g^v(X^h, Y^v) = g^v(X^v, Y^h) = 0,$
$g^v(X^v, Y^v) = g(X, Y) \circ \pi,\ \forall X, Y \in \mathcal{D}^1(N).$

Let now (f, g) be a *map*-structure on N i.e. $g \circ (f \times f) = g$ and $\widetilde{g} = g \circ (f \times I)$ its associated metric. Considering the lifts

61) $G = f^h + f^v,\ H = f^h - f^v,\ g_1 = g^h + g^v,\ g_2 = g^h - g^v,\ g_3 = \widetilde{g}^h + \widetilde{g}^v,\ g_4 = \widetilde{g}^h - \widetilde{g}^v,$

one obtains

Theorem 3.8. *If (f, g) is a map-structure, $\widetilde{g}$ its associated metric and ∇ a linear connection on N, then the tensor fields F, G, H, g_1 given by 59) and 61), determine on TN a mahp-structure with the associated metrics g_2, g_3, g_4.*

We remark that the pairs (f^h, g^h) and (f^v, g^v) determine *map*-structures on the subbundles HTN and VTN with $\widetilde{g}^h$ and $\widetilde{g}^v$ as associated metrics. After that the distributions (subbundles) W_α of the *ahp*-structure (F, G, H) are given by

62) $W_1 = (f^h)^+, \; W_2 = (f^h)^-, \; W_3 = (f^v)^+, \; W_4 = (f^v)^-.$

Considering local coordinates (x^i) on N and (x^i, y^i) on TN, where for $z = (x, y) \in T_xN$, $y = y^i \dfrac{\partial}{\partial x^i}$ and setting for ∇, $\nabla_{\frac{\partial}{\partial x^i}} \dfrac{\partial}{\partial x^j} = \Gamma_{ij}^k \dfrac{\partial}{\partial x^k}$, we obtain on TN the local vector fields and 1-forms

63) $\dfrac{\delta}{\delta x^i} = \left(\dfrac{\partial}{\partial x^i}\right)^h = \dfrac{\partial}{\partial x^i} - \Gamma_{ij}^k y^j \dfrac{\partial}{\partial x^k}, \; \delta y^i = (dx^i)^v = dy^i + \Gamma_{kj}^i y^j dx^k.$

Hence, to the natural basis $\left(\dfrac{\partial}{\partial x^i}\right)$ and co–basis (dx^i) on N, we can associate on TN the adapted basis $\left(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right)$ and co–basis $(dx^i, \delta y^i)$. In these bases, denoting by the same letter the matrix of each tensor field, we obtain

64) $F = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \; G = \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}, \; H = \begin{bmatrix} f & 0 \\ 0 & -f \end{bmatrix},$

$g_1 = \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}, \; g_2 = \begin{bmatrix} g & 0 \\ 0 & -g \end{bmatrix}, \; g_3 = \begin{bmatrix} \widetilde{g} & 0 \\ 0 & \widetilde{g} \end{bmatrix}, \; g_4 = \begin{bmatrix} \widetilde{g} & 0 \\ 0 & -\widetilde{g} \end{bmatrix}.$

Therefore, $\operatorname{tr} F = \operatorname{tr} G = 0$, $\operatorname{tr} H = 2\operatorname{tr} f$ and hence F and G determine always *apc*-stuctures on TN and G determines an *apc*-structure if and only if f determines an *apc*-structure on N. If g is a Riemannian metric on N, then (F, G, H, g_1) is a Riemannian *mahp*-structure on TN, g_1 being the Sasaki metric associated to metric g and the connection ∇ on N. The associated metrics g_2, g_3, g_4 are always of hyperbolic type. More precisely, g_2 and g_4 are always of neutral type and g_3 is neutral if and only if f is an *apc*-structure on N. In an orthogonal basis on N, formed by eigenvectors of f and the basis on TN formed by the corresponding horizontal and vertical lifts of these, the tensors of the Riemannian *mahp*-structure (F, G, H, g_1) and the associated metric g_2, g_3, g_4 are given by 18) and 21), where $n_1 = n_3$ and $n_2 = n_4$. Let now D be the *diagonal* lift of ∇, [3], i.e. the connection on TN given by

65) $D_{X^h} Y^h = (\nabla_X Y)^h, \; D_{X^h} Y^v = (\nabla_X Y)^v, \; D_{X^v} Y^h = D_{X^v} Y^v = 0, \quad \forall X, Y \in \mathcal{D}^1(N).$

For the horizontal and vertical lifts of $f \in \mathcal{D}_1^1(N)$ and $g \in \mathcal{D}_2^0(N)$, we obtain

66) $D_{X^h}(f^h) = (\nabla_X f)^h, \; D_{X^v}(f^h) = 0, \; D_{X^h}(f^v) = (\nabla_X f)^v, \; D_{X^v}(f^v) = 0;$

$D_{X^h}(g^h) = (\nabla_X g)^h, \; D_{X^v}(g^h) = 0, \; D_{X^h}(g^v) = (\nabla_X g)^v, \; D_{X^v}(g^v) = 0, \forall X \in \mathcal{D}^1(N).$

If (f, g) is a *map*-structure and $\widetilde{g}$ its associated metric, we get for the *mahp*-structure (F, G, H, g_1) and the associated metrics g_2, g_3, g_4,

$$\begin{aligned}
&DF = 0,\ D_{X^h}G = (\nabla_X f)^h + (\nabla_X f)^v,\ D_{X^v}G = 0,\\
&D_{X^h}H = (\nabla_X f)^h - (\nabla_X f)^v,\ D_{X^v}H = 0;\\
&D_{X^h}g_1 = (\nabla_X g)^h + (\nabla_X g)^v,\ D_{X^v}g_1 = 0,\\
&D_{X^h}g_2 = (\nabla_X g)^h - (\nabla_X g)^v,\ D_{X^v}g_2 = 0\\
&D_{X^h}g_3 = (\nabla_X \widetilde{g})^h + (\nabla_X \widetilde{g})^v,\ D_{X^v}g_3 = 0,\\
&D_{X^h}g_4 = (\nabla_X \widetilde{g})^h - (\nabla_X \widetilde{g})^v,\ D_{X^v}g_4 = 0,\ \forall X \in \mathcal{D}^1(N).
\end{aligned}$$

Hence, DF is always zero, DG and DH are simultaneous zero and namely when ∇f is zero. Dg_1 and Dg_2 are zero for $\nabla g = 0$ and Dg_3 and Dg_4 are zero for $\nabla \widetilde{g} = 0$. Resuming the previous considerations we obtain finally

Theorem 3.9. *If (f, g) is a map-structure on N and $\nabla = \psi_f(\nabla^g)$ is the canonical (f, g)-connection associated to it, then (F, G, H, g_1) is a mahp-structure on TN and the diagonal lift D of ∇ is compatible with this structure.*

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