25 Centro-affine geometry of complex hypersurfaces

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A complex centro-affine space C_n can be viewed as a real centro-affine space $\widetilde{C}_{2n}$ endowed with an automorphism J which satisfies the condition $J^2 = -I$. Then, the group of the automorphisms of C_n appears as the subgroup of the automorphisms of $\widetilde{C}_{2n}$ which commute with J. In addition a complex submanifold S_m in C_n appears as a real submanifold $\widetilde{S}_{2m}$ in $\widetilde{C}_{2n}$ whose tangent spaces at all points are invariant by J. Sometimes $\widetilde{S}_{2m}$ is called an invariant submanifold in $\widetilde{C}_{2n}$.

In particular, a complex hypersurface in C_n can be viewed as an invariant submanifofd of codimension 2 in $\widetilde{C}_{2n}$. Moreover, under some assumptions the plane spanned by the vector of position r and the vector $J(r)$ can be taken as normal space at every point of the hypersurface. Thus, the geometry of these hypersurfaces can be developed similarly to the geometry of real hypersurfaces in real centro-affine spaces [3]-[6]. However one seems that it has a more rich and more varied content.

In this work, our purpose is to sketch the theory of complex hypersurfaces in the complex centro-affine spaces. Further details will be given in a forthcoming paper. We shall use, with some exceptions, the terms and the notations from [5].

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Let $\widetilde{C}_{2n+2}$ be a real centro-affine space endowed with the complex structure J and let $\widetilde{S}_{2n}$ be a complex hypersurface given locally by the equation

$$r = r(u^i), \quad (i, j, k = 1, 2, ..., 2n). \tag{1}$$

We assume that the analytic vector function r satisfies the condition

$$(r, \widetilde{r}, r_1, ..., r_{2n}) \neq 0, \tag{2}$$

where $\widetilde{r} = J(r)$ and $r_i = \partial r / \partial u^i$.

Since $\widetilde{S}_{2n}$ is an invariant submanifold, we have also

$$\widetilde{r}_j = J(r_j) = F^i_j r_i, \tag{3}$$

from which, by applying again J one obtains

$$F^i_j F^j_k = -\delta^i_k, \tag{4}$$

i.e. F is the tensor field which defines the complex structure induced by J on $\widetilde{S}_{2n}$. Denoting by ρ (resp. $\widetilde{\rho}$) the covector of the tangent hyperplan which contains the tangent subspace to $\widetilde{S}_{2n}$ and the vector $\widetilde{r}$ (resp. r), at every point of $\widetilde{S}_{2n}$ we have

$$\rho r = 1,\ \rho\widetilde{r} = 0,\ \rho r_i = 0,\ \widetilde{\rho} r = 0,\ \widetilde{\rho}\widetilde{r} = 1,\ \widetilde{\rho} r_i = 0, \tag{5}$$

from which it follows

$$\rho = \frac{[\widetilde{r}, r_1, ..., r_{2n}]}{(r, \widetilde{r}, r_1, ..., r_{2n})},\quad \widetilde{\rho} = -\frac{[r, r_1, ..., r_{2n}]}{(r, \widetilde{r}, r_1, ..., r_{2n})}. \tag{6}$$

Also from (3) and (5) one obtains

$$\widetilde{\rho} = -J^*(\rho),\quad \rho\widetilde{r}_i = 0,\quad \widetilde{\rho}\widetilde{r}_i = 0. \tag{7}$$

Now, from (5) it results

$$\rho_j r_i + \rho \partial_j r_i = 0,\ \partial_j \rho_i r + \rho_i r_j = 0, \tag{8}$$

and the other ones which are obtained from these by replacing r by $\widetilde{r}$ or ρ by $\widetilde{\rho}$.

We assume in the sequel that $\widetilde{S}_{2n}$ is regular i.e. satisfies the condition

$$(\rho, \widetilde{\rho}, \rho_1, ..., \rho_{2n}) \neq 0. \tag{9}$$

By considering the centro-affine normalisation i.e. taking r and $\widetilde{r} = J(r)$ as normals, while ρ and $\widetilde{\rho} = -J^*(\rho)$ as conormals, we can write the fundamental equations

$$\begin{aligned} \partial_j r_i &= {}'\Gamma^k_{ji} r_k + {}'G_{ji} r + {}'H_{ji}\widetilde{r}, \\ \partial_j \rho_i &= {}''\Gamma^k_{ji} \rho_k + {}''G_{ji}\rho + {}''H_{ji}\widetilde{\rho}. \end{aligned} \tag{10}$$

It comes out that ${}'\Gamma$ and ${}''\Gamma$ are symmetric linear connections called the connections of the *first* and the *second kind*, respectively, while ${}'G, {}'H, {}''G, {}''H$ are symmetric tensor fields called *Eulerian* tensor fields, induced by the centro-affine normalisation of $\widetilde{S}_{2n}$. From (5), (7), (8) and (10) one obtains

$$\begin{aligned} {}'G_{ji} &= \rho\partial_j r_i = -\rho_j r_i = \partial_i \rho_j r = {}''G_{ij}, \\ {}'H_{ji} &= \widetilde{\rho}\partial_j r_i = -\widetilde{\rho} r_i = \partial_i \widetilde{\rho}_j r = -{}''H_{ij}, \end{aligned} \tag{11}$$

i.e. ${}'G = {}''G$ and ${}'H = -{}''H$.

Deleting the prime on G and H we can write the fundamental equations as follows

$${}'\nabla_j r_i = G_{ji} r + H_{ji}\widetilde{r},\quad {}''\nabla_j \rho_i = G_{ji}\rho - H_{ji}\widetilde{\rho}. \tag{12}$$

Differentiating covariantly with respect to ${}'\Gamma$ the both sides of the equality (3) and taking into account (12) we obtain

$${}'\nabla_k F^i_j = 0,\quad G_{kj} = H_{ki} F^i_j,\quad G_{kj} = -G_{ki} F^i_j. \tag{13}$$

The first relation tells us that $'\Gamma$ is a symmetric F-connection. The last two relations, multiplied with F give us

(14) $$G_{ki}F_h^kF_j^i = -G_{hj}, \quad H_{ki}F_h^kF_j^i = -H_{hj}$$

i.e. the quadratic forms G and H are skew-invariant with respect to F.

Form (5) and (14) we obtain '

(15) $$(\rho, \widetilde{\rho}, \rho_1, ..., \rho_{2n})(r, \widetilde{r}, r_1, ..., r_{2n}) = \det(G_{ij}).$$

So, $\widetilde{S}_{2n}$ is regular if and only if the tensor field G is nondegenerate.

Putting

$$\widetilde{\rho}_j = -J^*(\rho_j) = A_j^i\rho_i + B_j\rho$$

one obtains by (4), (5) and (11)

(16) $$\widetilde{\rho}_j = -F_j^i\rho_i$$

from which, by covariant differentiation with respect to $''\Gamma$ one obtains again the last two equalities from (13) as well as

(17) $$''\nabla_kF_j^i = 0.$$

This tells us that $''\Gamma$ is also an F-connection.

The equalities $G_{ji} = -\rho_jr_i$ and $H_{ji} = -\widetilde{\rho}_jr_i$ lead to

(18) $$\begin{aligned} \partial_kG_{ji} - {'\Gamma}_{kj}^hG_{hi} - {''\Gamma}_{ki}^hG_{jh} = 0, \\ \partial_kH_{ji} - {'\Gamma}_{kj}^hH_{hi} - {''\Gamma}_{ki}^hH_{jh} = 0. \end{aligned}$$

Therefore we have obtained

Proposition 1. *The pair of connections* $('\Gamma, ''\Gamma)$ *is conjugate* [5] *with respect to each or the tensor fields* G *and* H.

The integrability conditions of the fundamental equations (12) give us:

(19) $$\begin{aligned} {'R}_{kji}^h = G_{ki}\delta_j^h - G_{ji}\delta_k^h + H_{ki}F_j^h - H_{ji}F_k^h, \\ {'\nabla}_kG_{ji} = {'\nabla}_jG_{ki}, \quad {'\nabla}_kH_{ji} = {'\nabla}_jH_{ki}, \end{aligned}$$

(20) $$\begin{aligned} {''R}_{kji}^h = G_{ki}\delta_j^h - G_{ji}\delta_k^h + H_{ki}F_j^h - H_{ji}F_k^h, \\ {''\nabla}_kG_{ji} = {''\nabla}_jG_{ki}, \quad {''\nabla}_kH_{ji} = {''\nabla}_jH_{ki}. \end{aligned}$$

Therefore $'R = ''R$ and the tensor fields $'\nabla G, '\nabla H, ''\nabla G, ''\nabla H$ are completely symmetric.

Taking $k = h$ in (19_1) and (20_1) one obtains for the Ricci tensors of $'\Gamma$ and $''\Gamma$ the expression

(21) $$'R_{ji} = ''R_{ji} = -2(n-1)G_{ji}.$$

These equalities and (13) lead to

(22) $$G_{ji} = -\frac{1}{2(n-1)}\,{'R}_{ji}, \quad H_{ji} = \frac{1}{2(n-1)}\,{'R}_{jh}F_i^h,$$

which say that G and H are determined by F and the connection $'\Gamma$ or $''\Gamma$. As G is symmetric, it follows that the Ricci tensors $'R_{ji}$ and $''R_{ji}$ are symmetric, hence the both connections $'\Gamma$ and $''\Gamma$ are equiaffine.

By using (22) and (19_{1-2}) one obtains

$$
\begin{aligned}
&{}'R^h_{kji} = \frac{1}{2(n-1)}\,({}'R_{ji}\delta^h_k - {}'R_{ki}\delta^h_j + {}'R_{km}F^m_i F^h_j - {}'R_{jm}F^m_i F^h_k),\\
&{}'\nabla_k {}'R_{ji} = {}'\nabla_j {}'R_{ki}.
\end{aligned}
\tag{23}
$$

For any symmetric F-connection on a complex manifold of real dimension $2n$, the tensor field P of H-projective curvature is defined by:

$$
P^h_{kji} = R^h_{kji} + \delta^h_k P_{ji} - \delta^h_j P_{kj} - (P_{kj} - P_{jk})\delta^h_i + F^h_k Q_{ji} - F^h_j Q_{ki} - (Q_{kj} - Q_{jk})F^h_i,
\tag{24}
$$

where

$$
\begin{aligned}
&P_{ji} = \frac{1}{2(n+1)}\left[R_{ji} + \frac{1}{n-1}\,O^{hk}_{ji}(R_{hk} + R_{kh})\right],\\
&O^{hk}_{ji} = \frac{1}{2}\,(\delta^h_j\delta^k_i - F^h_j F^k_i),\quad Q_{ji} = -P_{jh}F^h_i.
\end{aligned}
\tag{25}
$$

When the connection is equiaffine, we have

$$
P_{ji} = -\frac{1}{2(n-1)}\,R_{ji}.
\tag{26}
$$

Hence

$$
P^h_{kji} = R^h_{kji} - \frac{1}{2(n-1)}\,(\delta^h_k R_{ji} - \delta^h_j R_{ki} + R_{km}F^m_i F^h_j - R_{jm}F^m_i F^h_k).
\tag{27}
$$

The F-connection Γ is H-projectively flat ($n \geq 2$) if and only if the tensor field P vanishes [2].

From (19), (22) and (23) it results

Proposition 2. *The connections of first and second kind, induced by the centro-affine normalisation on a complex hypersurface, are H-projective flat, equiaffine and symmetric F-connections.*

The fundamental equations (12), the integrability conditions (19) or (20) and the formulae (22) imply the following

Proposition 3. *An analytic manifold $\widetilde{S}_{2n}$, endowed with a complex structure F and an F-connection $'\Gamma$ (resp. $''\Gamma$) which is symmetric, equiaffine H-projectively flat, can be immersed as a complex, hypersurface in a complex centro-affine space $(\widetilde{C}_{2n+2}, J)$ such that F becomes the complex structure induced by J and $'\Gamma$ (resp. $''\Gamma$) be the induced linear connection of first kind (resp. second kind). The immersion is determined up to a complex centro-affine automorphism.*

Introducing the mean connection Γ and the deformation tensor h, for the pair of connections $('\Gamma, ''\Gamma)$ as follows

$$
\begin{aligned}
&\Gamma^k_{ji} = \frac{1}{2}\,({}'\Gamma^k_{ji} + {}''\Gamma^h_{ji}),\\
&h^k_{ji} = \frac{1}{2}\,({}'\Gamma^k_{ji} - {}''\Gamma^k_{ji})
\end{aligned}
\tag{28}
$$

we can write

$$'\Gamma^h_{ji} = \Gamma^k_{ji} + h^k_{ji},$$
$$''\Gamma^k_{ji} = \Gamma^k_{ji} - h^k_{ji}. \tag{29}$$

From (13) and (17) it follows

$$\nabla_k F^i_j = 0, \quad \nabla_k G_{ji} = 0, \quad \nabla_k H_{ji} = 0. \tag{30}$$

Thus we have

Proposition 4. *The pairs of tensors (F, G) and (F, H) determine on a complex hypersurface, complex metric structures* [1].

By means of the mean connection and of the deformation tensor, the fundamental equations become

$$\nabla_j r_i = h^k_{ji} r_k + G_{ji} r + H_{ji} \widetilde{r}, \quad \nabla_j \rho_i = -h^h_{ji} \rho_k + G_{ji} \rho - H_{ji} \widetilde{\rho}. \tag{31}$$

The integrability conditions of these giye us:

$$R^s_{kji} = \nabla_j h^s_{ki} - \nabla_k h^s_{ji} + h^m_{ki} h^s_{jm} - h^m_{ji} h^s_{km} + G_{ki} \delta^s_j - G_{ji} \delta^s_k + H_{ki} F^s_j - H_{ji} F^s_k, \tag{32}$$

$$h^m_{ji} G_{km} - h^m_{ki} G_{jm} = 0,$$
$$h^m_{ji} H_{km} - h^m_{ki} H_{jm} = 0, \tag{33}$$

$$R^s_{kji} = \nabla_k h^s_{ji} - \nabla_j h^s_{ki} + h^m_{ki} h^s_{jm} - h^m_{ji} h^s_{km} + G_{ki} \delta^s_j - G_{ji} \delta^s_k + H_{ki} F^s_j - H_{ji} F^s_k. \tag{34}$$

Putting

$$h_{ijk} = h^m_{ij} G_{km} \tag{35}$$

from (33_1) one obtains

$$h_{ijk} = h_{ikj}, \tag{36}$$

which says that the tensor field h is completely symmetric.

Then, from (33_2) it results

$$h_{ijm} F^m_k = h_{ikm} F^m_j. \tag{37}$$

By substracting and adding the equations (32) and (34) one obtains, respectively,

$$\nabla_k h^s_{ji} = \nabla_j h^s_{ki}, \tag{38}$$

$$R^s_{kji} = h^m_{ki} h^s_{jm} - h^m_{ji} h^s_{km} + G_{ki} \delta^s_j - G_{ji} \delta^s_k + G_{jh} F^h_i F^s_k - G_{kh} F^k_i F^s_j. \tag{39}$$

Putting

$$t_i = \frac{1}{2n}\, h^s_{is} \tag{40}$$

we obtains from (38)

$$\nabla_k t_j = \nabla_j t_k, \tag{41}$$

which says that the 1-form

$$t = t_j du^j \tag{42}$$

is closed.

Then, considering

$$\omega = (r, \widetilde{r}, r_1, ..., r_{2n}), \quad G = \det(G_{ij}) \tag{43}$$

we obtain

$$\begin{aligned} {}'\Gamma^h_{ih} &= \frac{\partial \ln |\omega|}{\partial u^i}, \\ {}''\Gamma^h_{ih} &= \frac{1}{2} \frac{\partial \ln |G|}{\partial u^i}, \end{aligned} \tag{44}$$

and from this

$$t_t = \frac{1}{2n} \frac{\partial}{\partial u^i} \ln \frac{|\omega|}{\sqrt{|G|}}, \tag{45}$$

i.e. the 1-form t is just exacte.

From the fundamental equations (31) and their integrability conditions, it follows

Proposition 5. *An analytic manifold $\widetilde{S}_{2n}$, endowed with a complex structure F, a tensor field G of type $(0,2)$ symmetric and nondegenerate and a tensor field h of type $(1,2)$ symmetric, which satisfy the equations (14_1), $30_{1-2})$, (33_1), (35), (38), and (39), can be immersed as a complex hypersurface in the complex centro-affine space $(\widetilde{C}_{2n+2}, J)$ such that F, G, h be, respectively, the complex structure and the tensor fields of type $(0,2)$ and $(1,2)$ induced be the centro-affine normalization. The immersion is determined up to a complex centro-affine automorphism.*

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