27 On Lie algebra bundles

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The structure of Lie algebra bundles is very important and much studied, in the last time, especially for its applications in the Theoretical Physics [1], [3]-[5]. The purpose of this work is to present some properties of derivations and linear connections, compatible with such a structure.

1. Definition of the Lie algebra bundle

For a paracompact and connected differentiable manifold M_n of class C^∞, let $\mathcal{F}(M)$ be the ring of real functions, $\mathcal{D}^r_s(M)$ the $\mathcal{F}(M)$–module of tensor fields of type (r,s) and $\mathcal{D}(M)$ the tensorial algebra. For a vector bundle $\xi = (E, \pi, M)$, we denote by $\mathcal{D}^r_s(M,E)$ the $\mathcal{F}(M)$–module of tensor fields of type (r,s) and by $\mathcal{D}(M,E)$ its tensorial algebra.

Definition 1. A *Lie algebra vector bundle* is a vector bundle $\xi = (E, \pi, M)$ equiped with a tensor field $C \in \mathcal{D}^1_2(M,E)$, which is skew–symmetric and satisfies the Jacobi identity, that is:

$$(1)\qquad \begin{gathered} C(u,v) + C(v,u) = 0, \\ C(u, C(v,w)) + C(v, C(w,u)) + C(w, C(u,v)) = 0, \\ \forall\, u,v,w \in \mathcal{D}^1_0(M,E). \end{gathered}$$

Setting for each $x \in M$ and $u_x, v_x \in E_x$,

$$(2)\qquad \{u_x, v_x\} = C_x(u_x, v_x),$$

we obtain a $\mathbb{R}$–Lie algebra structure on the fibre E_x, with C_x as structural tensor. Putting then, for $u, v \in \mathcal{D}^1_0(M,E)$

$$(3)\qquad \{u,v\} = C(u,v),$$

we obtain a $\mathcal{F}(M)$–Lie algebra structure on the module $\mathcal{D}^1_0(M,E)$, denoted by $\mathcal{D}^1_0(M,E,C)$. The vector field $\{u,v\}$ is called the *Lie product* or the *bracket* of u and v.

A Lie algebra vector bundle is called a *Lie algebra bundle* if there exists a $\mathbb{R}$–Lie algebra L and for each $x_0 \in M$ a vectorial chart $(\mathcal{U}, \varphi, L)$ such that for any $x \in \mathcal{U}$, the map $t_x : L \longrightarrow E_x$, given by $t_x(\vec{y}) = \varphi^{-1}(x, \vec{y})$, is a Lie algebra isomorphism. It follows that for a basis $\{\ell_a\} \subset L$,

$a = 1, 2, ..., r = rank\, L$ and the corresponding basis $\{e_a(x)\} = \{t_x(\ell_a)\} \subset E_x$, one obtains the structure equations

$$(4) \qquad \{e_b(x), e_c(x)\} = C^a_{bc} e_a(x),\ b, c = 1, 2, ..., r,$$

where C^a_{bc} are constant on $\mathcal{U}$. In this case all the fibres of ξ are isomorphic to L as Lie algebras.

Exemple. Considering the bundle of tensors of type $(1, 1)$ on ξ and setting

$$(5) \qquad [S, T] = S \circ T - T \circ S,\ \forall\, S, T \in \mathcal{D}^1_1(M, E),$$

one obtains on this bundle a structure of Lie algebra bundle with $g\ell(n : \mathrm{I\!R})$ as typical fibre.

2. Derivations compatible with the bracket

By a derivation in the vector bundle $\xi = (E, \pi, M)$, we shall understand a $\mathrm{I\!R}$–derivation in the tensorial algebra $\mathcal{D}(M, E)$ which preserves the type and commutes with the contractions. It is uniquely determined by its values on $\mathcal{F}(M)$ and $\mathcal{D}^1_0(M, E)$. The set of these derivations is denoted by $Der(M, E)$ and it is a $\mathcal{F}(M)$–module and a $\mathrm{I\!R}$–Lie algebra. For every derivation $D \in Der(M, E)$, we shall denote by $res^r_s(D)$ its restriction to $\mathcal{D}^r_s(M, E)$. In particular, $res^0_0(D)$ is a vector field on M. There is an isomorphism between the algebra $\mathcal{D}^1_1(M, E)$ and the subalgebra of the drivations on ξ with the restriction to $\mathcal{D}^0_0(M, E) = \mathcal{F}(M)$ equal to zero. This isomorphism associates to each $S \in \mathcal{D}^1_1(M, E)$ the derivation $D = i(S)$ given by $i(S)(f) = 0,\ f \in \mathcal{F}(M)$ and $i(S)(u) = S(u),\ u \in \mathcal{D}^1_0(M, E)$. If $\xi = (E, \pi, M, C)$ is a Lie algebra vector bundle, then setting for each $u \in \mathcal{D}^1_0(M, E, C)$

$$(6) \qquad ad_u(v) = \{u, v\},$$

it comes out that ad_u is a tensor field of type $(1, 1)$ on ξ and that the map $ad : \mathcal{D}^1_0(M, E, C) \longrightarrow \mathcal{D}^1_1(M, E)$ is a $\mathcal{F}(M)$–Lie algebra morphism. The kernel of ad is the center K of $\mathcal{D}^1_0(M, E, C)$ and its image is an ideal in $\mathcal{D}^1_1(M, E)$, which is isomorphic to the factor algebra $\mathcal{D}^1_0(M, E, C)/K$.

Definition 2. One says that a derivation $D \in Der(M, E)$ is *compatible* with the bracket on the Lie algebra vector bundle $\xi = (E, \pi, M, C)$ if it is a derivation in the $\mathrm{I\!R}$–Lie algebra $\mathcal{D}^1_0(M, E, C)$.

In other words $D \in Der(M, E)$ is compatible with the bracket on ξ if and only if

$$(7) \qquad D\{u, v\} = \{Du, v\} + \{u, Dv\},\ \forall\, u, v \in \mathcal{D}^1_0(M, E).$$

It follows from here

Proposition 1. *A derivation $D \in Der(M, E)$ is compatible with the bracket on the Lie algebra vector bundle $\xi = (E, \pi, M, C)$ if and only if the derivative of the structural tensor field C is zero,*

$$(8) \qquad DC = 0.$$

Remark 1. The set $Der(M, E, C)$ of the derivations compatible with the bracket on ξ is a $\mathcal{F}(M)$–submodule and a $\mathrm{I\!R}$–Lie subalgebra in $Der(M, E)$.

Remark 2. The set $i(\mathcal{D}^1_1(M, E, C)$ of the derivations of the form $D = i(S)$, with $S \in \mathcal{D}^1_1(M, E)$, compatible with the bracket on ξ, is a $\mathcal{F}(M)$–submodule and a subalgebra of $Der(M, E, C)$.

Remark 3. All derivations of the form $D = i(ad_u)$ with $u \in \mathcal{D}^1_0(M,E)$, called inner derivations, are compatible with the bracket on ξ and their set $In(\mathcal{D}^1_0(M,E,C)) = Im(i \circ ad)$ is a $\mathcal{F}(M)$–submodule and an ideal in $Der(M,E,C)$.

Exemple. Every derivation on the vector bundle $\xi = (E, \pi, M)$ induces a derivation compatible with the bracket on the Lie algebra bundle of tensors of type $(1,1)$ on ξ.

3. Linear connections compatible with the bracket

A linear connection ∇ in the bundle $\xi = (E, \pi, M)$ may be defined as a map $\nabla : \mathcal{D}^1_0(M) \longrightarrow Der(M,E)$ which is $\mathcal{F}(M)$–linear and satisfies the condition

$$res^0_0 \circ \nabla = 1_{\mathcal{D}^1_0(M)}. \tag{9}$$

To each linear connection ∇ one associates the map $R : \mathcal{D}^1_0(M) \times \mathcal{D}^1_0(M) \times \mathcal{D}^1_0(M,E) \longrightarrow \mathcal{D}^1_0(M,E)$ given by

$$R_{XY}u = [\nabla_X, \nabla_Y]u - \nabla_{[X,Y]}u, \quad \forall X, Y \in \mathcal{D}^1_0(M),\ u \in \mathcal{D}^1_0(M,E) \tag{10}$$

and called the *curvature* of the connection ∇. The set of linear connections in the bundle ξ is a $\mathcal{F}(M)$–affine module [2], which we denote by $\mathcal{C}(M,E)$. Considering the exact sequence of $\mathcal{F}(M)$–modules

$$0 \longrightarrow \mathcal{D}^1_1(M,E) \xrightarrow{\ i\ } Der(M,E) \xrightarrow{res^0_0} \mathcal{D}^1_0(M) \longrightarrow 0, \tag{11}$$

a linear connection ∇ in the vector bundle ξ is a right splitting for this sequence. It follows from here

Proposition 2. *Given a linear connection ∇ in the vector bundle $\xi = (E, \pi, M)$, every derivation $D \in Der(M,E)$ can be decomposed uniquely as follows*

$$D = i(S) + \nabla_X, \tag{12}$$

where $X = res^0_0(D)$ and $S = res^1_0(D - \nabla_X)$.

Hence, the connection ∇ determines a decomposition of the $\mathcal{F}(M)$–module $Der(M,E)$ in the direct sum

$$Der(M,E) = i(\mathcal{D}^1_1(M,E) \oplus \nabla(\mathcal{D}^1_0(M)). \tag{13}$$

Considering two derivations $D_1, D_2 \in Der(M,E)$ and setting

$$D_1 = i(S_1) + \nabla_{X_1}, \quad D_2 = i(S_2) + \nabla_{X_2}, \tag{14}$$

one obtains for their bracket,

$$[D_1, D_2] = i([S_1, S_2] + \nabla_{X_1}S_2 - \nabla_{X_2}S_1 + R_{X_1X_2}) + \nabla_{[X_1,X_2]}. \tag{15}$$

This formula suggests us to consider the $\mathcal{F}(M)$–linear map

$$res^1_1 \circ \nabla : \mathcal{D}^1_0(M) \longrightarrow Der\,\mathcal{D}^1_1(M,E)$$

and from the identity

$$[\nabla_X, \nabla_Y]S - \nabla_{[X,Y]}S = [R_{XY}, S], \; X, Y \in \mathcal{D}_0^1(M), \; S \in \mathcal{D}_1^1(M, E), \tag{16}$$

we obtain

Proposition 3. *The map $res_1^1 \circ \nabla$ is a Lie algebra morphism if and only if*

$$R = \alpha \otimes I, \tag{17}$$

where α is a 2–form on M and I is the identical automorphism of ξ.

In this case, we can consider the semidirect product of R–Lie algebras

$$\mathcal{D}_1^1(M, E) \underset{res_1^1 \circ \nabla}{\times} \mathcal{D}_0^1(M),$$

defining the bracket by

$$[(S_1, X_1), (S_2, X_2)] = ([S_1, S_2] + \nabla_{X_1} S_2 - \nabla_{X_2} S_1, [X_1, X_2]). \tag{18}$$

Then, from the formula (15), we obtain

Proposition 4. *The map $i \times \nabla : \mathcal{D}_1^1(M, E) \underset{res_1^1 \circ \nabla}{\times} \mathcal{D}_0^1(M) \longrightarrow Der(M, E)$, given by*

$$(i \times \nabla)(S, X) = i(S) + \nabla_X, \tag{19}$$

is a Lie algebra isomorphism if and only if the connection ∇ has zero curvature.

Remark 4. Only in this case, the connection ∇ is a splitting of the sequence (11) regarded as a Lie algebras exact sequence.

Definition 3. One says that a linear connection ∇ in the Lie algebra vector bundle $\xi = (E, \pi, M, C)$ is *compatible* with the bracket if for each $X \in \mathcal{D}_0^1(M)$ the derivation ∇_X is in $Der(M, E, C)$.

From Proposition 1 it follows

Proposition 5. *The linear connection ∇ in the Lie algebra vector bundle $\xi = (E, \pi, M, C)$ is compatible with the bracket if and only if*

$$\nabla_X C = 0, \quad \forall X \in \mathcal{D}_0^1(M). \tag{20}$$

For this condition we may give the following geometrical characterization

Proposition 6. *The linear connection ∇ in the Lie algebra vector bundle ξ is compatible with the bracket if and only if for each curve on the manifold M, the parallel transport, defined by ∇, establishes a Lie algebra isomorphism between the fibres of ξ along the curve.*

Using the partition of unity and the formula (4) one obtains

Proposition 7. *Every Lie algebra bundle admits a linear connection compatible with the bracket.*

Conversely, from Proposition 6, it follows that any Lie algebra vector bundle which admits a linear connection, compatible with the bracket, is a Lie algebra bundle.

The set of these connections is a $\mathcal{F}(M)$–affine submodule of $\mathcal{C}(M,E)$ denoted by $\mathcal{C}(M,E,C)$.

Exemple. Every linear connection in a vector bundle $\xi=(E,\pi,M)$ induces a linear connection, compatible with the bracket, in the Lie algebra bundle of tensors of type $(1,1)$ on ξ.

Remark. If ∇ is a linear connection compatible with the bracket, in the Lie algebra bundle $\xi=(E,\pi,M,C)$, then from the formula (17) it follows that the sequence of $\mathcal{F}(M)$–modules

$$0\longrightarrow \mathcal{D}_1^1(M,E,C)\xrightarrow{\ i\ } Der(M,E,C)\xrightarrow{res_0^0}\mathcal{D}_0^1(M)\longrightarrow 0 \tag{21}$$

is an exact one and that every linear connection $\nabla'\in\mathcal{C}(M,E,C)$ is a right splitting of this sequence.

If $\nabla\in\mathcal{C}(M,E,C)$ and $S\in\mathcal{D}_1^1(M,E,C)$, then from

$$\nabla_X S=[\nabla_X,S], \tag{22}$$

it follows that $\nabla_X S\in\mathcal{D}_1^1(M,E,C)$. Hence from Proposition 4, one obtains

Proposition 8. *The map*

$$i\times\nabla:\mathcal{D}_1^1(M,E,C)\underset{res_1^1\circ\nabla}{\times}\mathcal{D}_0^1(M)\longrightarrow Der(M,E,C),$$

where $\nabla\in\mathcal{C}(M,E,C)$, is a Lie algebra isomorphism if and only if the curvature of ∇ is zero.

Let now D_1 and D_2 be two derivations of the form

$$D_1=i(ad_{u_1})+\nabla_{X_1},\quad D_2=i(ad_{u_2})+\nabla_{X_2}, \tag{23}$$

where $\nabla\in\mathcal{C}(M,E,C)$. Then, we have from (15)

$$[D_1,D_2]=i\left(ad_{(\{u_1,u_2\}+\nabla_{X_1}u_2-\nabla_{X_2}u_1)}+R_{X_1X_2}\right)+\nabla_{[X_1,X_2]}. \tag{24}$$

Considering the map $res_0^1\circ\nabla:\mathcal{D}_0^1(M)\longrightarrow Der\,\mathcal{D}_0^1(M,E,C)$, from (10) it follows that it is a Lie algebra morphism if and only if$R=0$. In this case we can consider the semidirect product of Lie algebras $\mathcal{D}_0^1(M,E,C)\underset{res_0^1\circ\nabla}{\times}\mathcal{D}_0^1(M)$ with the bracket defined by

$$[(u_1,X_1),(u_2,X_2)]=(\{u_1,u_2\}+\nabla_{X_1}u_2-\nabla_{X_2}u_1,[X_1,X_2]). \tag{25}$$

Then, from the formula (24) it follows

Proposition 9. *The map*

$$(i\circ ad)\times\nabla:\mathcal{D}_0^1(M,E,C)\underset{res_0^1\circ\nabla}{\times}\mathcal{D}_0^1(M)\longrightarrow Der(M,E,C)$$

given by

$$((i\circ ad)\times\nabla)(u,X)=i(ad_u)+\nabla_X, \tag{26}$$

where $\nabla\in\mathcal{C}(M,E,C)$ is a Lie algebra isomorphism if and only if∇ has zero curvature.

The kernel of the morphism $(i \circ ad) \times \nabla$ is $K \times \{0\} \subset \mathcal{D}_0^1(M, E, C) \times \mathcal{D}_0^1(M)$. Considering the factor Lie algebra $\mathcal{D}_0^1(M, E, C)/K$ and setting

$$\overline{\nabla}_X \bar{u} = \overline{\nabla_X u}, \quad \forall \bar{u} \in \mathcal{D}_0^1(M, E, C)/K, \tag{27}$$

one obtains a derivation on this algebra.

Then, taking the map $\overline{\nabla} : \mathcal{D}_0^1(M) \longrightarrow Der(\mathcal{D}_0^1(M, E, C)/K)$ given by the rule $X \longrightarrow \overline{\nabla}_X$, from the formula

$$[\overline{\nabla}_X, \overline{\nabla}_Y]\bar{u} - \overline{\nabla}_{[X,Y]}\bar{u} = \overline{R_{XY}u} = \overline{0}, \tag{28}$$

it follows that it is a Lie algebras morphism. Hence, we can consider the semidirect product of Lie algebras $\mathcal{D}_0^1(M, E, C)/K \underset{\overline{\nabla}}{\times} \mathcal{D}_0^1(M)$, with the bracket

$$[(\bar{u}_1, X_1), (\bar{u}_2, X_2)] = \left(\{\bar{u}_1, \bar{u}_2\} + \overline{\nabla}_{X_1}\bar{u}_2 - \overline{\nabla}_{X_2}\bar{u}_1, [X_1, X_2]\right). \tag{29}$$

Setting

$$ad^*_{\bar{u}}(\bar{v}) = \overline{ad_u v}, \quad \forall \bar{u}, \bar{v} \in \mathcal{D}_0^1(M, E, C)/K. \tag{30}$$

it results

Proposition 10. *If there is on the Lie algebra bundle $\xi = (E, \pi, M, C)$ a linear connection ∇ of zero curvature, which is compatible with the bracket, then the map*

$$((i \circ ad^*) \times \nabla) : \mathcal{D}_0^1(M, E, C)/K \underset{\overline{\nabla}}{\times} \mathcal{D}_0^1(M) \longrightarrow Im((i \circ ad) \times \nabla)$$

is a Lie algebras isomorphism.

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