37. Para-Hermitian and para-Kähler manifolds

Collaboration with P.M. Gadea and J. Muñoz Masqué

Quaderni dell'Istituto di Matematica
Facoltà di Economia e Commercio
Università di Messina
no. 1, 1995, 1-70.

Contents

1 Introduction

As is well-known, there is, up to isomorphisms, three algebras of dimension 2 on $\mathbb{R}$, i.e., $\mathbb{C} = \mathbb{R}[x]/(x^2+1)$, $\mathbb{A} = \mathbb{R}[x]/(x^2-1)$ and $\mathbb{D} = \mathbb{R}[x]/(x^2)$. The theory of complex functions is richer than the other two. Algebraically, this reflects the fact that $\mathbb{C}$ is a field, while $\mathbb{A}$ and $\mathbb{D}$ are not: $\mathbb{A}$ has divisors of zero and $\mathbb{D}$ has even nilpotent elements. Nevertheless, until relatively a few years ago we did not have – with Grotendhieck and his school, and others – a systematic treatment of the geometries over rings with divisors of zero and even nilpotent elements, and so the importance of the theory of functions with values in $\mathbb{A}$ or $\mathbb{D}$ cannot be given as being definitively established. Furthermore, from the structural point of view, it is clear that each of the three earlier algebras determines a perfectly defined geometry, on an equal footing with the others. Moreover, it is not reasonable to demand that the theory of functions on $\mathbb{A}$ or $\mathbb{D}$ have the same importance as that on $\mathbb{C}$ as a prerequisite for its study. With this mentality, also the passage from $\mathbb{R}$ to $\mathbb{C}$ would be criticizable. Actually, some of this mentality did exist at the end of the XVIIIth century and the beginning of the XIXth, since, as is well-known, complex geometry had serious problems in its development, and only the importance of its results, clearing – and sometimes relativizing – the nature of real geometry, dissipated the doubts of a good number of "real" geometers. On the other hand, it must be said that each geometry has its own characteristics, different from the other geometries, which is the fact justifying its study. Only the actual development of a geometry justifies its existence. In the case of paracomplex geometry, the problem has been perhaps mainly of terminology, because the geometrical richness of its elements (existence of two real distributions, of the underlying foliations, associated almost product structure, etc., which in the complex case are not "real" elements) has frequently permitted to attack the problems from many different points of view, forgetting to maintain a common methodology for all of them. When one reads papers on distributions, foliations, almost product structures, hyperbolic geometry, etc., it is very frequent that there exists a lexic of paracomplex geometry that could unify all of those works in a subject only, whose development cause and develops all the questions treated "ad hoc" by the authors. And this is another of the characteristics of a genuine geometry: to unite under a common lexic different situations, guiding the intuition and formulating its own problems.

The present paper is an updated and expanded version of [53]. Here, we shall also understand by paracomplex geometry the geometry related to the algebra of paracomplex numbers and, mainly, the study of the structures on differentiable manifolds called paracomplex structures. Moreover, when we consider a compatible neutral pseudo-Riemannian metric, we have the para-Hermitian structures, para-Kähler structures and their variants.

Acknowledgements. We are very grateful to Professors E. Reyes Iglesias, A. Bejancu, F. Etayo, P. Fortuny, A. Montesinos Amilibia and J. A. Oubiña for their valuable suggestions.

2 Paracomplex numbers

2.1 Algebraic theory

Let us consider the real plane $\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$, endowed with its natural structure of vector space over $\mathbb{R}$ and with its canonical basis $\{e_1 = (1,0), e_2 = (0,1)\}$. If $z = (x,y)$, $z' = (x',y')$, are elements of $\mathbb{R}^2$, defining a product $zz' = (xx'+yy', xy'+yx')$, one obtains a new composition law in $\mathbb{R}^2$ which determines, with the addition and the multiplication by real numbers, a structure of associative, commutative and unitary algebra over $\mathbb{R}$ of rank 2, denoted by $\mathbb{A}$ and called *algebra of paracomplex numbers*.

This algebra contains the subfield $\mathbb{R}'$ of elements $(x,0)$, and the map $x \mapsto (x,0)$ is an isomorphism between $\mathbb{R}$ and $\mathbb{R}'$. Identifying 1 with (1,0) and letting $j = (0,1)$, every number in $\mathbb{A} \equiv \mathbb{R}^2$ can be written as $z = (x,y) = x(1,0) + y(0,1) = x + jy$ with $j^2 = 1$. The useful bases for $\mathbb{A}$ are $\{1, j\}$ and $\{e^+ = \frac{1}{2}(1+j), e^- = \frac{1}{2}(1-j)\}$. In the second one every paracomplex number z can be written as $z = z^+e^+ + z^-e^-$, $z^+, z^- \in \mathbb{R}$. To every paracomplex number $z = x + jy$ we can associate its *conjugate* $\bar{z} = x - jy$, and the map $z \mapsto \bar{z}$ is an automorphism of $\mathbb{A}$. Every paracomplex number $z = x + jy$ has an associated real number, its *modulus* $|z|$, defined by $|z| = \sqrt{|z\bar{z}|} = \sqrt{|x^2 - y^2|}$. If $|z| \neq 0$, then there exists the inverse, defined by $z^{-1} = \pm\bar{z}/|z|^2$, according to $x^2 - y^2 > 0$ or < 0, respectively. The set $\tilde{\mathbb{A}}$ of invertible paracomplex numbers is a multiplicative group isomorphic to $\mathbb{R}^* \times \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$. The isomorphism is given by $z = z^+e^+ + z^-e^- \mapsto (z^+, z^-)$. The elements $z \neq 0$ in $\mathbb{A}$ such that $|z| = 0$ ($z\bar{z} = 0$) are the divisors of zero, which have the expression $z = x(1 \pm j)$ in the basis $\{1, j\}$ and $z = xe^{\pm}$ in the basis $\{e^+, e^-\}$.

Associating to each paracomplex number $z = x + jy$ the point in the plane $\mathbb{R}^2$ with coordinates (x, y), we obtain the *paracomplex plane*. In this plane there are some relevant elements: the x-axis or *real axis*, the y-axis or *paraimaginary axis*, the hyperbola $x^2 - y^2 = 1$ of paracomplex numbers such that $z\bar{z} = 1$, the conjugate hyperbola $x^2 - y^2 = -1$ of paracomplex numbers such that $z\bar{z} = -1$, the diagonal lines $x + y = 0$ and $x - y = 0$, i.e., the divisors of zero. For the geometry of the paracomplex plane see [224].

We can also associate to each paracomplex number $z = x + jy$ the real matrix S given by

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

and so obtain an isomorphism between $\mathbb{A}$ and the algebra of such matrices. Notice that $z\bar{z} = \det S$. In another way, we can associate to each paracomplex number $z = z^+e^+ + z^-e^-$ the matrix

$$\begin{pmatrix} z^+ & 0 \\ 0 & z^- \end{pmatrix}$$

and so we have an isomorphism of $\mathbb{A}$ with the corresponding matrix algebra.

We recall here, as an application of paracomplex numbers, the construction of a pseudo-Riemannian Cayley transformation when working with pseudo-Riemannian Poincaré models of constant curvature, given in [96]. Notice that paracomplex numbers permit us to work in any signature of the metric.

2.2 Paraholomorphic functions

Consider $\mathbb{A}$ endowed with the usual topology of $\mathbb{R}^2$, an open subset U of $\mathbb{A}$ and a paracomplex function $f\colon U \longrightarrow \mathbb{A}$. We shall suppose that all functions are at least of class C^1. Putting $z = x + jy$ and $f(z) = P(x,y) + jQ(x,y)$, one can prove that both the limit and the continuity for the function f can be expressed in terms of the limit and the continuity of the real functions P and Q. We shall say that the function $f\colon U \subset \mathbb{A} \longrightarrow \mathbb{A}$ is a *paraholomorphic function at* $z_0 \in U$ if and only if there exists $\alpha \in \mathbb{A}$ such that

$$f(z_0 + \Delta z) - f(z_0) = \alpha\Delta z + \varepsilon(z_0, \Delta z)||\Delta z||\,,$$

where $||\Delta z|| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ and $\varepsilon(z_0, \Delta z) \to 0$ when $\Delta z \to 0$. It follows that f is paraholomorphic at z_0 if and only if P and Q are differentiable at (x_0, y_0) and satisfy the conditions

$$\frac{\partial P}{\partial x}(z_0) = \frac{\partial Q}{\partial y}(z_0), \qquad \frac{\partial P}{\partial y}(z_0) = \frac{\partial Q}{\partial x}(z_0)\,. \tag{2.1}$$

The number $\dfrac{df}{dz}(z_0) = \alpha$ will be called *the derivative of f at z_0*, and we have (dropping now the subscript z_0)

$$\frac{df}{dz} = \frac{\partial P}{\partial x} + j\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} + j\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} + j\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} + j\frac{\partial P}{\partial y}\,.$$

The function f is said to be *paraholomorphic in U* if it is paraholomorphic at every $z \in U$. Writing $z = z^+e^+ + z^-e^-$ and $f = f^+e^+ + f^-e^-$, one can see that f is paraholomorphic in U if and only if in U: f^+ and f^- are differentiable, f^+ depends only of z^+ and f^- depends only of z^-. Thus we have

$$\frac{\partial f^+}{\partial z^-} = \frac{\partial f^-}{\partial z^+} = 0 \qquad \text{and} \qquad \frac{df}{dz} = \frac{\partial f^+}{\partial z^+}e^+ + \frac{\partial f^-}{\partial z^-}e^-\,.$$

We shall say that f is *paraholomorphic of class C^k* if it has derivatives up to the order k and the derivative f^k is continuous. One can easily see that f is paraholomorphic of class C^k if and only if the real functions P and Q are of class C^k and also satisfy the conditions (2.1). The function f is said to be *analytic at $z_0 \in U$* if there exists a neighbourhood $V_{z_0} \subset U$ such that f admits a convergent power series expansion

$$f(z) = \sum_{k=0}^{\infty} \lambda_k (z - z_0)^k, \qquad z \in V_{z_0}, \quad \lambda_k \in \mathbb{A}\,.$$

The function f is analytic if and only if the real functions P and Q are analytic and satisfy (2.1). Writing $f = f^+e^+ + f^-e^-$, one can prove that f is analytic if and only if f^+ and f^- are real analytic functions of z^+ and z^-, respectively. The concepts of paraholomorphic and analytic functions – which are not equivalent [98] – can be immediately extended to the case of several paracomplex variables.

The interest of paraholomorphic functions is greater than it can seem at a first glance, as the following three considerations show:

(1) To begin with, we recall some definitions and results from [99]. For the terms *almost paracomplex manifold* and the related ones, see §3 in the present paper.

Definition 2.1. A map $f\colon (M, J) \to (M', J')$ between almost paracomplex manifolds of class C^∞ (resp. of class C^ω) is said to be a *morphism of almost paracomplex manifolds* if for all $x \in M$ we have $f_* \circ J_x = J'_{f(x)} \circ f_*$.

Proposition 2.2. *Let (M, J) be an almost paracomplex manifold of class C^∞ (resp. of class C^ω). A necessary and sufficient condition in order the map $F\colon M \to \mathbb{A}$ to be a morphism of almost paracomplex manifolds is that there exist functions $f, g \in C^\infty(M)$ (resp. $f, g \in C^\omega(M)$) such that:*

(i) $F = (1 + j)f + (1 - j)g$,

(ii) *f and g are first integrals of $T^+(M)$ and $T^-(M)$, respectively; that is to say, $X^+(f) = 0$, $X^-(g) = 0$, for all $X^+ \in \Gamma(T^+(M))$, $X^- \in \Gamma(T^-(M))$, where $T^+(M)$ (resp. $T^-(M)$) denotes the eigenbundle on M corresponding to the eigenvalue $+1$ (resp. -1) of J.*

Definition 2.3. The morphisms of an almost paracomplex manifold (M, J) of class C^∞ in $\mathbb{A}$ are named *almost paraholomorphic functions* of (M, J).

We are now in a position to explain that the interest of paraholomorphicity is global, since paraholomrphic functions are narrowly linked to the properties of the foliations defined by a paracomplex structure. In other words, the ring $C^{aph}(M)$ of almost paraholomorphic functions measures the integrability of the distributions defined by $T^+(M)$ and $T^-(M)$. Let us first introduce some definitions aiding to exactly determine the above assertion. Let $\mathcal{D}$ be a distribution on M. Let us define by recurrence the distribution $\mathcal{D}^{(k)}$ in the following way: $\mathcal{D}^{(1)} = [\mathcal{D}, \mathcal{D}]$, $\mathcal{D}^{(k)} = [\mathcal{D}, \mathcal{D}^{(k-1)}]$, $k = 1, 2, \ldots$. Then $\mathcal{D}$ is said to be *completely non integrable* if for every $x \in M$ there exists an integer k such that the values of the vector fields of $\mathcal{D}^{(k)}$ at x span $T_x(M)$. It is easily proved by induction that the first integrals of $\mathcal{D}$ are also annihilated by all the fields of $\mathcal{D}^{(k)}$. Consequently, if $\mathcal{D}$ is completely non integrable and M is connected, the only first integrals of $\mathcal{D}$ are the constant functions. Hence, from Proposition 2.2 it follows that if the distributions defined by both $T^+(M)$ and $T^-(M)$ are completely non integrable and M is connected, then the only almost paraholomorphic functions of M are the elements of $\mathbb{A}$.

At the opposite end we have the case where both distributions are completely integrable, which we shall now consider. We first recall that a *paracomplex manifold* of class C^∞ (resp. of class C^ω) is an almost paracomplex manifold (M, J) of class C^∞ (resp. of class C^ω) such that the distributions defined by the subbundles $T^+(M)$ and $T^-(M)$ associated to the structure are both involutive.

Definition 2.4. [99]. The morphisms of a paracomplex manifold of class C^∞ in $\mathbb{A}$ will be named *paraholomorphic functions.*

Theorem 2.5. [99] *A $2n$-dimensional almost paracomplex manifold (M, J) of class C^∞ is paracomplex if and only if its sheaf of germs of almost paraholomorphic functions is locally isomorphic to the sheaf of germs of paraholomorphic functions of $\mathbb{A}^n$.*

Moreover, if the distributions defined by the subbundles $T^+(M)$ and $T^-(M)$ are both involutive, then the sheaf of germs of paraholomorphic functions is locally isomorphic to the sheaf of germs of paraholomorphic functions on $\mathbb{A}^n$, where $\dim M = 2n$.

(2) The existence of "many" global paraholomorphic functions is also linked to the *immersion problem*; that is, to the possibility of embedding a paracomplex manifold (M, J) into $\mathbb{A}^N$, for some sufficiently great N. This is the paracomplex analog of the question in theory of complex manifolds which gives rise to Stein manifolds.

(3) On the other hand, let us suppose that for a given paracomplex manifold M there exist the quotient manifolds $M^+ = M/\mathcal{F}^+(M)$ and $M^- = M/\mathcal{F}^-(M)$, being $\mathcal{F}^+(M)$ and $\mathcal{F}^-(M)$ the foliations corresponding to $T^+(M)$ and $T^-(M)$ respectively; i.e., that each topological quotient space admits a structure of differentiable manifold such that the canonical projections p^+ and p^- of M on M^+ and M^-, respectively, are submersions. Under these conditions, the map $f = p^+ \times p^- \colon M \to M^+ \times M^-$ is a paraholomorphic immersion with respect to the the canonical paracomplex structure of $M^+ \times M^-$. Consequently, the paracomplex manifolds having quotient manifolds with regard to the distributions defined by the eigenbundles can be obtained in the way just described. If M is compact (and connected), its structure is that defined in a finite covering by the canonical structure of the product. If, in particular, M is simply connected, then $M = M^+ \times M^-$. Thus, every paracomplex differentiable manifold admitting the quotient submanifolds M^+ and M^-, admits a subordinate structure of analytic manifold.

Remark 2.6. Some relations between harmonic maps f of Lorentz surfaces M into pseudo-Riemannian manifolds and paraholomorphic quadratic differentials on M (and other objects) associated with f are given in [72].

More specifically, let $f \colon (M, h) \to (N, g)$ be a smooth non-degenerate mapping from the Lorentz surface (M, h) into the pseudo-Riemannian manifold (N, g). The *tension field* $\mathcal{T}(f)$ ([72, p. 9]) decomposes into

$$\mathcal{T}(f) = \mathcal{H}^h(f) + \nu^h(f),$$

where $\nu^h(f) \in E = df(TM)$ and $\mathcal{H}^h(f) = E^\perp$ are the tangential and normal components of $\mathcal{T}(f)$, respectively. The vector field $\mathcal{H}^h(f)$ is said to be the *h-mean curvature vector field along f.*

On the other hand, let $d^H f$ be the extension by paracomplex linearity of the differential map df and let $\partial/\partial z = \frac{1}{2}\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)$ for given h-isothermal coordinates (x, y) on M. Then, for $q = g(D_{d^H f(\partial/\partial z)}, D_{d^H f(\partial/\partial z)})$, where D denotes the Levi-Civita connection on N, the (globally defined) quadratic differential $Q = q\,dz^2$ associated with f is said to be *paraholomorphic* if q is paraholomorphic.

Among other results in [72], the author obtains the following one:

Theorem 2.7.

(i) *The quadratic differential* $Q = q\,dz^2$ *associated to* f *is paraholomorphic if and only if* $\nu^h(f) = 0$.

(ii) f *is harmonic if and only if* Q *is paraholomorphic and* $\mathcal{H}(f) = 0$.

2.3 Paracomplex modules

Let $V^{\mathbb{A}}$ be a commutative group $(V, +)$, endowed with a structure of unitary module over the ring $\mathbb{A}$ of paracomplex numbers. Let $V^{\mathbb{R}}$ denote the group $(V, +)$, endowed with the structure of real vector space inherited from the restriction of scalars to $\mathbb{R}$. We shall call $V^{\mathbb{R}}$ the *real model of* $V^{\mathbb{A}}$ or the *real vector space associated to* $V^{\mathbb{A}}$. Putting

$$J(u) = ju, \quad P^+(u) = e^+u, \quad P^-(u) = e^-u, \qquad u \in V^{\mathbb{A}},$$

we obtain

$$\begin{gathered} J^2 = 1_V, \quad P^{+2} = P^+, \quad P^{-2} = P^-, \\ P^+ \circ P^- = P^- \circ P^+ = 0, \\ P^+ + P^- = 1_V, \quad P^+ - P^- = J. \end{gathered}$$

Hence, J defines a product structure on $V^{\mathbb{R}}$, and P^+, P^- are the associated supplementary projection operators, $P^+ = (1/2)(1_V + J)$ and $P^- = (1/2)(1_V - J)$. Writing then $V^+ = P^+(V)$ and $V^- = P^-(V)$, it follows that $V^{\mathbb{R}} = V^{\mathbb{R}+} \oplus V^{\mathbb{R}-}$, and that $V^{\mathbb{R}+}$ and $V^{\mathbb{R}-}$ are the eigenspaces of J corresponding to the eigenvalues $+1$ and -1, respectively. A vector $u \in V^{\mathbb{A}}$, $u \neq 0$, is said to be a *singular vector* if and only if there exists $\lambda \neq 0$ such that $\lambda u = 0$. As it is easily seen, a vector is singular if and only if it is an eigenvector for J.

Suppose $V^{\mathbb{A}}$ has paracomplex dimension n, and let $\mathcal{B}^{\mathbb{A}} = \{e_1, \ldots, e_n\}$ be a basis. We then have for $u \in V^{\mathbb{A}}$,

$$u = \sum_{k=1}^{n} z_k e_k\,.$$

Writing now $z_k = x_k + jy_k$, we obtain

$$u = \sum_{k=1}^{n} x_k e_k + \sum_{k=1}^{n} y_k e_{n+k}\,,$$

where $e_{n+k} = Je_k$. It follows that $\mathcal{B}^{\mathbb{R}} = \{e_k, e_{n+k}\}$ is a basis of $V^{\mathbb{R}}$ and thus we have $\dim V^{\mathbb{R}} = 2n$. We shall call $\mathcal{B}^{\mathbb{R}}$ the *real basis of the first kind associated to* $\mathcal{B}^{\mathbb{A}}$. For $z_k = z_k^+ e_k^+ + z_k^- e_k^-$, we have

$$u = \sum_{k=1}^{n} z_k^+ e_k^+ + \sum_{k=1}^{n} z_k^- e_k^- ,$$

where $e_k^+ = P^+ e_k = e^+ e_k$ and $e_k^- = P^- e_k = e^- e_k$. One can see that $\mathcal{B}^{\mathbb{R}+} = \{e_k^+\}$ and $\mathcal{B}^{\mathbb{R}-} = \{e_k^-\}$ are basis for $V^{\mathbb{R}+}$ and $V^{\mathbb{R}-}$, respectively, and that $\mathcal{B}^{\mathbb{R}\pm} = \{e_k^+, e_k^-\}$ is a basis for $V^{\mathbb{R}}$. We shall say that $\mathcal{B}^{\mathbb{R}\pm}$ is the *real basis of the second kind associated to* $\mathcal{B}^{\mathbb{A}}$. Consequently, $V^{\mathbb{R}+}$ and $V^{\mathbb{R}-}$ are real subspaces of the same dimension n, and thus the eigenvalues ± 1 of J have the same multiplicity n, i.e., $\operatorname{Tr} J = 0$.

Conversely, let $V_{2n}^{\mathbb{R}}$ be a $2n$-dimensional real vector space endowed with a product structure J, that is, with $J^2 = 1$, such that $\operatorname{Tr} J = 0$. If we define on the underlying abelian group $(V, +)$ the multiplication by paracomplex numbers as

$$(a + jb)u := au + bJ(u) ,$$

the space V becomes a unitary paracomplex module of dimension n, which we denote $V_n^{\mathbb{A}}$, for which $V_{2n}^{\mathbb{R}}$ is the associated real vector space. This is the reason we call *paracomplex structure* on $V_{2n}^{\mathbb{R}}$ to its traceless involutive automorphism J.

An $\mathbb{A}$-linear transformation T on V is also $\mathbb{R}$-linear and satisfies $T \circ J = J \circ T$. Conversely, an $\mathbb{R}$-linear transformation on V commuting with J is also $\mathbb{A}$-linear. Thus, we have the group isomorphism

$$GL(V_n^{\mathbb{A}}) \approx GL_J(V_{2n}^{\mathbb{R}}) = \{T \in GL(V_{2n}^{\mathbb{R}}) : T \circ J = J \circ T\} .$$

Let us consider a basis $\mathcal{B}^{\mathbb{A}} = \{e_k\}$ for $V_n^{\mathbb{A}}$. Writing $T(e_k) = \sum_{l=1}^{n} \alpha_k^l e_l$ it follows that the map $T \mapsto \alpha = (\alpha_l^k)$ defines an isomorphism $GL(V_n^{\mathbb{A}}) \approx GL(n, \mathbb{A})$. Putting $\alpha_l^k = a_l^k + j b_l^k$, we have $T(e_k) = \sum_{l=1}^{n} a_k^l e_l + \sum_{l=1}^{n} b_k^l e_{n+l}$, $T(e_{n+k}) = \sum_{l=1}^{n} b_k^l e_l + \sum_{l=1}^{n} a_k^l e_{n+l}$. Consequently, the map

$$T \mapsto \begin{pmatrix} a & b \\ b & a \end{pmatrix} ,$$

where $a = (a_l^k)$, $b = (b_l^k)$, establishes an isomorphism

$$GL_J(V_{2n}^{\mathbb{R}}) \approx GL_{J_0}(2n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) : AJ_0 = J_0 A\} ,$$

where

$$J_0 = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} .$$

Finally, putting $\alpha_l^k = \alpha_l^{+k} e_+ + \alpha_l^{-k} e_-$, we have $T(e_k^+) = \sum_{l=1}^{n} \alpha_k^{+l} e_l^+$, $T(e_k^-) = \sum_{l=1}^{n} \alpha_k^{-l} e_l^-$, and so the basis $\mathcal{B}^{\mathbb{R}\pm}$ originates an isomorphism

$$GL_J(V_{2n}^{\mathbb{R}}) \approx GL_{\tilde{J}}(2n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) : A\tilde{J} = \tilde{J}A\} ,$$

where

$$\tilde{J} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} .$$

It is easy to see that we have isomorphisms

$$GL(V_n^{\mathbb{A}}) \approx GL(V_{2n}^{\mathbb{R}}) \approx GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) .$$

2.4 Para-Hermitian modules

A para-Hermitian form over the unitary paracomplex module $V_n^{\mathbb{A}}$ is a map $h\colon V_n^{\mathbb{A}} \times V_n^{\mathbb{A}} \longrightarrow \mathbb{A}$, which is $\mathbb{A}$-linear in the first argument and para-Hermitian symmetric, i.e.:

$$h(u_1+u_2,v) = h(u_1,v)+h(u_2,v), \quad h(\lambda u,v) = \lambda h(u,v), \quad h(v,u) = \overline{h(u,v)}\,.$$

Writing

$$h(u,v) = g(u,v) + j\varphi(u,v)\,,$$

we obtain two real bilinear forms g and φ on $V_{2n}^{\mathbb{R}}$, which satisfy

$${}^t g = g, \quad {}^t\varphi = -\varphi, \quad g\circ(J\times J) = -g\,,$$

$$\varphi\circ(J\times J) = -\varphi, \quad g\circ(1_V\times J) = \varphi, \quad \varphi\circ(1_V\times J) = g\,.$$

Conversely, given on $(V_{2n}^{\mathbb{R}}, J)$ a symmetric bilinear form g satisfying $g\circ(J\times J) = -g$, taking $\varphi = g\circ(1_V\times J)$ and $h = g + j\varphi$, it follows that h is a para-Hermitian form on $V_n^{\mathbb{A}}$. The form h is non-degenerate if and only if g (or φ) is. We shall denote by $(V_n^{\mathbb{A}}, h)$ the *para-Hermitian module* – with h non-degenerate – and by $(V_{2n}^{\mathbb{R}}, g, J)$ the associated real vector space, endowed with the structure (g, J), called *para-Hermitian structure* on $V_{2n}^{\mathbb{R}}$.

The $\mathbb{A}$-linear transformations of $V_n^{\mathbb{A}}$ preserving the metric h constitute a group $U_h(V_n^{\mathbb{A}})$, called *the paraunitary group on* $(V_n^{\mathbb{A}}, h)$. We have a group isomorphism

$$U_h(V_n^{\mathbb{A}}) \approx GL_{J,g}(V_{2n}^{\mathbb{R}}) = \{T\in GL(V_{2n}^{\mathbb{R}}) : T\circ J = J\circ T,\ g\circ(T\times T) = g\}\,.$$

The form φ defines a symplectic structure on $V_{2n}^{\mathbb{R}}$, which is also preserved by the elements $T\in GL_{J,g}(V_{2n}^{\mathbb{R}})$. By considering a basis $\mathcal{B}^{\mathbb{A}} = \{e_k\}$ for $V_n^{\mathbb{A}}$, we have $h_{kl} = g_{kl} + j\varphi_{kl}$, and for $u = \sum_{k=1}^n z_k e_k = \sum_{k=1}^n x_k e_k + \sum_{k=1}^n y_k e_{n+k}$, $v = \sum_{k=1}^n \zeta_k e_k = \sum_{k=1}^n \xi_k e_k + \sum_{k=1}^n \eta_k e_{n+k}$, we obtain

$$h(u,v) = {}^t z h\bar\zeta = {}^t U G V + j\,{}^t U\Phi V\,,$$

where $z=(z_k)$, $x=(x_k)$, $y=(y_k)$, $\zeta=(\zeta_k)$, $\xi=(\xi_k)$, $\eta=(\eta_k)$,

$$U = \begin{pmatrix} x\\ y\end{pmatrix},\ V = \begin{pmatrix} \xi\\ \eta\end{pmatrix},\ G = \begin{pmatrix} g & -\varphi\\ \varphi & -g\end{pmatrix},\ \Phi = \begin{pmatrix} \varphi & -g\\ g & -\varphi\end{pmatrix},$$

and $g = (g_{kl})$, $\varphi = (\varphi_{kl})$. One can take [224] an orthonormal basis $\mathcal{B} = \{e_i\}$, and then have $h_{kl} = \delta_{kl}$, $g_{kl} = \delta_{kl}$, $\varphi_{kl} = 0$. So, in this basis one has

$$h(u,v) = {}^t z\bar\zeta, \quad g(u,v) = {}^t x\xi - {}^t y\eta, \quad \varphi(u,v) = -{}^t x\eta + {}^t y\xi\,, \tag{2.2}$$

the associated matrices being

$$h_0 = I_n, \quad G_0 = \begin{pmatrix} I_n & 0\\ 0 & -I_n\end{pmatrix}, \quad \Phi_0 = \begin{pmatrix} 0 & -I_n\\ I_n & 0\end{pmatrix}.$$

Remark 2.8.

(1) The quadratic form g has necessarily signature (n,n) and so it determines a neutral structure on $V_{2n}^{\mathbb{R}}$.

(2) The restrictions of the bilinear forms g and φ to $V^{\mathbb{R}+}$ and $V^{\mathbb{R}-}$ are null, and thus these subspaces are maximally isotropic for g and φ.

The orthonormal basis $\mathcal{B}$ in $(V^{\mathbb{A}}, h)$ and its associated basis of the first kind in (V^{2n}, g, J), establish the isomorphisms

$$U_h(V_n^{\mathbb{A}}) \approx U(n, \mathbb{A}) = \{\alpha \in GL(n, \mathbb{A}) : {}^t\alpha\bar{\alpha} = I_n\}$$

and

$$GL_{J,g}(V_{2n}^{\mathbb{R}}) \approx GL_{J_0,G_0}(2n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) : AJ_0 = J_0A,\ {}^tAG_0A = G_0\}.$$

If h is given by (2.2) with respect to the orthonormal basis $\mathcal{B}$ of $V_n^{\mathbb{A}}$, taking the associated basis of the second kind in $V_{2n}^{\mathbb{R}}$ we have

$$u = \sum_{k=1}^{n} z_k^+ e_k^+ + \sum_{k=1}^{n} z_k^- e_k^-, \qquad v = \sum_{k=1}^{n} \zeta_k^+ e_k^+ + \sum_{k=1}^{n} \zeta_k^- e_k^-,$$

and hence

$$x = \frac{z^+ + z^-}{2}, \quad y = \frac{z^+ - z^-}{2}, \quad \xi = \frac{\zeta^+ + \zeta^-}{2}, \quad \eta = \frac{\zeta^+ - \zeta^-}{2},$$

from which

$$g(u,v) = \frac{1}{2}({}^tz^+\zeta^- + {}^t\zeta^+ z^-), \qquad \varphi(u,v) = \frac{1}{2}({}^tz^+\zeta^- - {}^t\zeta^+ z^-),$$

and the associated matrices are

$$\tilde{G} = \frac{1}{2}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \qquad \tilde{\Phi} = \frac{1}{2}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The basis $\mathcal{B}^{\mathbb{R}\pm}$ establishes a group isomorphism

$$GL_{J,g}(V_{2n}^{\mathbb{R}}) \approx GL_{\tilde{J},\tilde{g}} = \left\{\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} : A \in GL(n, \mathbb{R})\right\}.$$

Hence $U_h(V_n^{\mathbb{A}})$ is isomorphic with $GL(n, \mathbb{R})$.

2.5 Paracomplex projective spaces

Let $V^{\mathbb{A}}$ be a paracomplex unitary module, $V^{\mathbb{R}}$ the associated real vector space, $V^{\mathbb{A}*} = V^{\mathbb{A}} - \{0\}$, and $\tilde{\mathbb{A}} = \{z \in \mathbb{A} : |z| \neq 0\}$ the multiplicative group of invertible paracomplex numbers. Let us denote by $\sim$ the equivalence relation in $V^{\mathbb{A}*}$ given by $u \sim v \iff \exists\, z \in \tilde{\mathbb{A}} : v = zu$. The corresponding quotient space will be denoted

$$P(V^{\mathbb{A}}) = V^{\mathbb{A}*}/\tilde{\mathbb{A}},$$

and will be called the *paracomplex projective space* associated to $V^{\mathbb{A}}$. We can consider in a similar way the real projective space associated to $V^{\mathbb{R}}$, denoted by $P(V^{\mathbb{R}}) = V^{\mathbb{R}*}/\mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$. In order to see the relation between $P(V^{\mathbb{A}})$ and $P(V^{\mathbb{R}})$, we must see which are the respective equivalence classes. For every $u \in V^{\mathbb{A}*}$, its equivalence class $[u]^{\mathbb{A}} = \{zu : z \in \tilde{\mathbb{A}}\} = \tilde{\mathbb{A}}u$, is contained into the submodule $\mathbb{A}u$ of $V^{\mathbb{A}}$. If u is singular, that is, $u \in (V^+ \cup V^-) - \{0\}$, then every vector in $\mathbb{A}u$ is $\mathbb{A}$-linearly dependent, and consequently $\mathbb{A}u$ is not free. A point $[u]^{\mathbb{A}}$ corresponding to a singular vector u will be called a *singular point* of $P(V^{\mathbb{A}})$. The set of singular points is the reunion of the projective subspaces $P(V^{\mathbb{A}+})$ and $P(V^{\mathbb{A}-})$ of $P(V^{\mathbb{A}})$. If u is singular, then either $u = u^+ \in V^{\mathbb{A}+}$ or $u = u^- \in V^{\mathbb{A}-}$, and writing $z = z^+e^+ + z^-e^-$, we obtain $zu = z^+u$ or $zu = z^-u$.

Hence, to the submodule $\mathbb{A}u$ corresponds the real vector line $\mathbb{R}u$ belonging either to the subspace $V^{\mathbb{R}+}$ or to the subspace $V^{\mathbb{R}-}$. It follows that to the singular point $[u]^{\mathbb{A}}$ corresponds in $P(V^{\mathbb{R}})$ the point $[u]^{\mathbb{R}} = \mathbb{R}^* u$, contained into the real projective subspace $P(V^{\mathbb{R}+})$ or $P(V^{\mathbb{R}-})$. These subspaces will be called the *axes*, or the *absolute* or *invariant subspaces* of $P(V^{\mathbb{R}})$. If the vector u is not singular, then $u = u^+ + u^-$, with $u^+, u^- \neq 0$, and $zu = z^+u^+ + z^-u^-$. It follows that $\mathbb{A}u$ is a 1-dimensional submodule of $V^{\mathbb{A}}$, to which it corresponds in $V^{\mathbb{R}}$ the subspace $\mathbb{R}u^+ \oplus \mathbb{R}u^-$, which has real dimension 2, and intersects $V^{\mathbb{R}+}$ and $V^{\mathbb{R}-}$, respectively, in the vector lines $\mathbb{R}u^+$ and $\mathbb{R}u^-$. Consequently, to a nonsingular point $[u]^{\mathbb{A}}$ of $P(V^{\mathbb{A}})$ corresponds in $P(V^{\mathbb{R}})$ the line intersecting $P(V^{\mathbb{R}+})$ and $P(V^{\mathbb{R}-})$ in the singular points $[u^+]^{\mathbb{R}}$ and $[u^-]^{\mathbb{R}}$, respectively. We can conclude that $P(V^{\mathbb{A}})$ decomposes into two subsets:

(1) the subset of singular points $P(V^{\mathbb{A}+}) \cup P(V^{\mathbb{A}-})$, which is in bijective correspondence with the set $P(V^{\mathbb{R}+}) \cup P(V^{\mathbb{R}-})$, that is, with the axes of $P(V^{\mathbb{R}})$,

(2) the subset $P(\tilde{V}^{\mathbb{A}}) = \tilde{V}^{\mathbb{A}}/\tilde{\mathbb{A}}$, where $\tilde{V}^{\mathbb{A}} = V^{\mathbb{A}} - \{V^{\mathbb{A}+} \cup V^{\mathbb{A}-}\}$, of nonsingular points, which is in bijective correspondence with the set of lines of $P(V^{\mathbb{R}})$ which lay upon the axes $P(V^{\mathbb{R}+})$ and $P(V^{\mathbb{R}-})$. These lines constitute a congruence called *absolute* or *invariant congruence.*

If one associates to each line in this congruence the couple of singular points upon which it lays, a bijection $P(\tilde{V}^{\mathbb{A}}) \longleftrightarrow P(V^{\mathbb{R}+}) \times P(V^{\mathbb{R}-})$ is obtained. We can conclude that

$$P(V^{\mathbb{A}}) \approx \{P(V^{\mathbb{R}+}) \times P(V^{\mathbb{R}-})\} \bigcup \{P(V^{\mathbb{R}+}) \cup P(V^{\mathbb{R}-})\} .$$

For $V^{\mathbb{A}} = \mathbb{A}^{n+1}$, the set $P_n(\mathbb{A}) = \tilde{\mathbb{A}}^{n+1}/\tilde{\mathbb{A}}$ has been defined by Rozenfeld [223], [224, p. 578] and Libermann [144] as the *paracomplex projective space.* It is topologically equivalent to $P_n(\mathbb{R}) \times P_n(\mathbb{R})$ (see [144]). Libermann-Rozenfeld's definition is the best one from a geometrical point of view, as they consider the nonsingular *paracomplex lines.* But our present definition of the paracomplex projective space as $P(V^{\mathbb{A}}) = V^{\mathbb{A}*}/\tilde{\mathbb{A}}$, has the advantage of containing all the algebraically reasonable equivalence classes, and also that its real model is the whole real projective space $P(V^{\mathbb{R}}) = V^{\mathbb{R}*}/\mathbb{R}^*$, with the involution $\mathbf{j}$ defined by $\mathbf{j}([u]^{\mathbb{R}}) = [Ju]^{\mathbb{R}}$, where J is the paracomplex structure on $V^{\mathbb{R}}$. For $V^{\mathbb{A}} = \mathbb{A}^2$, the real model $(P_3(\mathbb{R}), \mathbf{j})$ has been defined by O. Mayer [150], who named it the *hyperbolic biaxial space.* For $V^{\mathbb{A}} = \mathbb{A}^n$, the real model $(P_{2n-1}(\mathbb{R}), \mathbf{j})$ has been named by Norden and others [251] the *hyperbolic biaffine space.* The paracomplex projective model $P_n(\mathbb{B})$ [87] (see Section 8.2) is an open subset in the fourfold covering space $\tilde{\mathbb{A}}^{n+1}/\tilde{\mathbb{A}}_{++}$ over Libermann-Rozenfeld's projective space, being $\mathbb{A}_{++}$ the connected component of the unity in the group $\tilde{\mathbb{A}}$.

We recall here an application of paracomplex numbers: a way of interpreting a projective geometry as an elliptic geometry. Consider pairs of points x, y and pairs of hyperplanes α, β in $P_n(\mathbb{R})$. Such quadruples possess a projective invariant $(x,\beta)(y,\alpha)/(x,\alpha)(y,\beta)$. Combine x and α into one $A = xe^+ + \alpha e^-$. Analogously, $B = ye^+ + \beta e^-$. A and B may be considered as points of the paracomplex projective n-space $P(\mathbb{A}^{n+1})$; indeed, multiplication of A with the paracomplex number $z^+e^+ + z^-e^-$ yields $z^+xe^+ + z^-\alpha e^-$. Hence, it means multiplication of x and α separately. Using the conjugation of paracomplex numbers one gets $A\bar{A} = (x, \alpha)$, $B\bar{B} = (y, \beta)$, $A\bar{B} \cdot B\bar{A} = (x,\beta)(y,\alpha)$. So the above projective invariant of four elements may be written as an invariant of two points $A\bar{B} \cdot B\bar{A}/A\bar{A} \cdot B\bar{B}$, which provides the paracomplex projective space with an elliptic structure. This device, systematically used by Rozenfeld [222]-[224], as Freudenthal [85] points out, can be applied to projective geometry over complex numbers or quaternions, and also to symplectic geometry, by considering paracomplex (split) quaternions.

3 Paracomplex manifolds

3.1 Some definitions and results

In order to obtain a better understanding of the ideas and results in the survey, we shall now recall some general definitions concerning (almost) paracomplex, (almost) para-Hermitian and (almost) para-Kähler manifolds. From now on, all the manifolds and geometric objects are C^∞.

Definition 3.1. An *almost product structure* J on a differentiable manifold M is a $(1,1)$ tensor field J on M such that $J^2 = 1$. The pair (M, J) is called an *almost product manifold.* An *almost paracomplex manifold* is an almost product manifold (M, J) such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of J, respectively, have the same rank. (Note that the dimension of an almost paracomplex manifold is necessarily even.) Equivalently, a splitting of the tangent bundle TM of a differentiable manifold M, into the Whitney sum of two subbundles $T^\pm M$ of the same fiber dimension is called an *almost paracomplex structure on* M. An almost paracomplex structure on a $2n$-dimensional manifold M may alternatively be defined as a G-structure on M with structural group $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$.

A *paracomplex manifold* is an almost paracomplex manifold (M, J) such that the G-structure defined by the tensor field J is integrable. An integrable almost product manifold is usually called a *locally product manifold.* Thus, a paracomplex manifold is a locally product manifold (M, J) such that if the characteristic polynomial of J is $(x-1)^r(x+1)^s$, $r + s = \dim M$, then $r = s$. We can give another – equivalent – definition of paracomplex manifold in terms of local homeomorphisms in the space $\mathbb{A}^n$ and paraholomorphic changes of charts, in a way similar to the complex case.

A definition of a paracomplex structure by considering a vector bundle E on a manifold M is given in [21].

Proposition 3.2. *An almost paracomplex manifold (M, J) is paracomplex if and only if it satisfies one of the following equivalent conditions:*

(1) *The two distributions defined on M by J are involutive.*

(2) *The Nijenhuis tensor N of J, defined by*

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y], \qquad X, Y \in \mathfrak{X}(M),$$

vanishes everywhere.

(3) *There exists a torsionless linear connection parallelizing J.*

Proposition 3.3. ([127]) *Let (M, J) be a $2n$-dimensional paracomplex manifold. Then M has an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ with U_α open and*

$$\varphi_\alpha = (x_1^\alpha, \dots, x_n^\alpha, y_1^\alpha, \dots, y_n^\alpha),$$

a coordinate map satisfying the following condition: if $U_\alpha \cap U_\beta \neq \emptyset$, then the para-Cauchy-Riemann *equations*

$$\frac{\partial x_k^\beta}{\partial x_l^\alpha} = \frac{\partial y_k^\beta}{\partial y_l^\alpha}, \quad \frac{\partial x_k^\beta}{\partial y_l^\alpha} = \frac{\partial y_k^\beta}{\partial x_l^\alpha}, \qquad 1 \leq k, l \leq n, \quad \alpha, \beta \in A, \tag{3.1}$$

hold. In this case, on each U_α, J is given by

$$J\frac{\partial}{\partial x_k^\alpha} = \frac{\partial}{\partial y_k^\alpha}, \quad J\frac{\partial}{\partial y_k^\alpha} = \frac{\partial}{\partial x_k^\alpha}. \tag{3.2}$$

Conversely, suppose that M has an atlas $\{(U_\alpha, \varphi_\alpha)\}$ satisfying (3.1)*. Then, if we define J on U_α by* (3.2)*, then J is globally defined on M, and (M, J) is a paracomplex manifold.*

Definition 3.4. Let (M, J) and (M', J') be (almost) paracomplex manifolds. Then a smooth map f of M to M' is called a *paraholomorphic map* if the relation $f_{*p} \circ J_p = J'_{f(p)} \circ f_{*p}$ is satisfied for each point $p \in M$, where f_{*p} is the differential of f at p. If there is a paraholomorphic diffeomorphism of M onto M', then (M, J) and (M', J') are said to be paraholomorphically equivalent. A paraholomorphic diffeomorphism of M onto itself is called a paraholomorphic transformation of M. We denote by $\mathrm{Aut}(M, J)$ the group of paraholomorphic transformations of M. (Notice that the eigenbundles of J are invariant under paraholomorphic transformations.)

Let M be a real differentiable manifold and let $\mathcal{F}_{\mathbb{A}}(M)$ be the $\mathbb{A}$-algebra of differentiable functions of M on $\mathbb{A}$. We can define a *paracomplex vector field* on M in the same way as we do in the real and complex case; that is, as a derivation of $\mathcal{F}_{\mathbb{A}}(M)$. We denote by $\mathfrak{X}_{\mathbb{A}}(M)$ the module of paracomplex vector fields on M. Given $Z \in \mathfrak{X}_{\mathbb{A}}(M)$, we define $Z|_p$, $p \in M$, by $Z|_p(f) = (Z(f))(p)$. The *conjugate* of a paracomplex vector field Z is the paracomplex vector field $\bar{Z}$ defined by $\bar{Z}(f) = \overline{Z(\bar{f})}$. Any $Z \in \mathfrak{X}_{\mathbb{A}}(M)$ can be written in a unique way as $Z = X + jY$, where $X, Y \in \mathfrak{X}(M)$. Moreover, if $\{e_k\}$ is a basis of $\mathfrak{X}(M)$, then it is also a basis of $\mathfrak{X}_{\mathbb{A}}(M)$. The Lie bracket of paracomplex vector fields is defined as usual.

The way of construction of the graded ring $\bigwedge_{\mathbb{A}}(M) = \bigoplus_k \bigwedge_{\mathbb{A}}^k(M)$ of *paracomplex (differentiable) forms* is similar to the real case: $\bigwedge_{\mathbb{A}}^k(M)$ is the $\mathcal{F}_{\mathbb{A}}(M)$-module of alternate k-linear applications from $\mathfrak{X}_{\mathbb{A}}(M) \times \ldots \times \mathfrak{X}_{\mathbb{A}}(M)$ (k times) onto $\mathcal{F}_{\mathbb{A}}(M)$. For $k = 0$ we define $\bigwedge_{\mathbb{A}}^0(M) = \mathcal{F}_{\mathbb{A}}(M)$. As to paracomplex vector fields, a paracomplex form can be written as $\omega = \alpha + j\beta$, where α, β are real forms. The exterior differentiation of paracomplex forms is defined as usual. It is a real and commutative endomorphism d of $\bigwedge_{\mathbb{A}}(M)$ of degree 1 and zero square such that $df(X) = Xf$, $\quad f \in \mathcal{F}_{\mathbb{A}}(M)$, $Z \in \mathfrak{X}_{\mathbb{A}}(M)$.

3.2 Examples of paracomplex manifolds

(1) ([55]) The product manifold $M^n \times M^n$ of a real manifold by itself has a canonical paracomplex structure.

(2) ([127, 144]) Let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be the natural coordinates on $\mathbb{R}^{2n}$. Let us consider the following two kinds of foliations: $x_k + y_k = \text{const}$, and $x_k - y_k = \text{const}$, $1 \le k \le n$, which define a paracomplex structure on $\mathbb{R}^{2n}$. These foliations are invariant under translations by the lattice $\mathbb{Z}^{2n}$ of all integral points in $\mathbb{R}^{2n}$. So, they naturally induce a paracomplex structure on the torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$.

(3) ([130]) The product manifold $M \times M'$ of two almost paracontact manifolds M and M' admits an almost paracomplex structure.

(4) ([168]) The group $G = SL(2, \mathbb{R})$ acts on its Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ by conjugation. As the invariant bilinear form $(X, Y) = \frac{1}{2}\,\mathrm{Tr}\, XY$ on $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ has signature (2,1), Ad defines a double covering of $SL(2, \mathbb{R})$ onto $SO_0(2, 1)$. Choose the basis

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \; Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, and use the corresponding coordinates $(x, y, z) \leftrightarrow xX + yY + zZ$ to identify $\mathfrak{g}$ with $\mathbb{R}^3$. The G-orbits in $\mathfrak{g}$ are of several types. The *hyperbolic orbits*, defined by $G \cdot (\lambda X) = G \cdot (\lambda Y)$, $\lambda > 0$, are diffeomorphic to the hyperbola $G \cdot (\lambda X) \approx Q_{+\lambda} = \{(x, y, z) : x^2 + y^2 - z^2 = \lambda^2\}$. These orbits are pseudo-Riemannian symmetric spaces, where the corresponding involution σ is given by conjugation by X

$$\sigma\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix},$$

and the corresponding fixpoint group is

$$H = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R} - \{0\}\right\}.$$

The tangent space of M at $(\lambda, 0, 0) = \lambda X$ is now identified with $T_{(\lambda,0,0)}Q_{+\lambda} = \mathbb{R}Y \oplus \mathbb{R}Z = \mathfrak{m}$, and the involution $(a, b) \mapsto (b, a)$ defined by $\frac{1}{2}\,\mathrm{ad}_{\mathfrak{m}}X$ is a paracomplex structure commuting with $\mathrm{Ad}(H)$ making $Q_{+\lambda}$ a paracomplex manifold.

(5) Many authors, among which C. Bejan, V. Cruceanu, S. Ianuş, S. Ishihara, T. Nagano, V. Oproiu, R. Rosca, C. Udriste and K. Yano, have considered almost paracomplex structures on the tangent bundle of a manifold M. Let ∇ be a linear connection on M and denote by X^v and X^h the vertical and horizontal lift respectively to the tangent bundle TM ([260]) of the vector field $X \in \mathfrak{X}(M)$. Putting then

$$P(X^v) = X^v, \quad P(X^h) = -X^h, \quad Q(X^v) = X^h, \quad Q(X^h) = X^v, \tag{3.3}$$

they obtain two almost paracomplex structures on TM. The structure P is paracomplex if and only if ∇ has vanishing curvature, and Q is paracomplex if and only if ∇ has both vanishing torsion and curvature. These structures have been extended to the case of a nonlinear connection, and to the specific cases of a nonlinear connection defined by a Finsler, Lagrange or Hamilton structure ([29, 38, 39, 175]). Similar structures for the cotangent bundle are obtained from a connection ∇ and a non-degenerate (0,2) tensor field g on M [52]. If α is a differentiable 1-form and X a vector field on M, α^v denotes the vertical lift of α and X^h the horizontal lift of X to T^*M, putting

$$P(X^h) = -X^h, \quad P(\alpha^v) = \alpha^v, \quad Q(X^h) = (X^\flat)^v, \quad Q(\alpha^v) = (\alpha^\sharp)^h, \tag{3.4}$$

where $\flat$ and $\sharp$ are the g-musical isomorphisms, they obtain two almost paracomplex structures on T^*M. P is paracomplex if ∇ has vanishing curvature and Q is paracomplex if and only if both the exterior covariant differential Dg of g given by

$$(Dg)(X, Y) = \nabla_X(Y^\flat) - \nabla_Y(X^\flat) - [X, Y]^\flat$$

and the curvature of ∇ vanish. The case where ∇ is symmetric has been considered in [21]. In this reference one can find more examples.

4 Para-Hermitian manifolds

4.1 Almost para-Hermitian manifolds

Definition 4.1. An *almost para-Hermitian manifold* (M, g, J) is a differentiable manifold M endowed with an almost product structure J and a pseudo-Riemannian metric g, compatible in the sense that

$$g(JX, Y) + g(X, JY) = 0, \quad X, Y \in \mathfrak{X}(M). \tag{4.1}$$

An *almost para-Hermitian structure* on a differentiable manifold M is a G-structure on M whose structural group is the real representation of the paraunitary group $U(n, \mathbb{A})$ given at the end of subsection (2.4). An almost para-Hermitian manifold can also be defined as a differentiable manifold with an almost para-Hermitian structure. It is easy to check that an almost para-Hermitian manifold is necessarily almost paracomplex and that the metric g has signature (n, n).

A *para-Hermitian manifold* is a manifold with an integrable almost para-Hermitian structure (g, J). That is, the G-structure associated to J is integrable [86].

Given an almost para-Hermitian manifold (M, g, J), we shall call *fundamental* 2-*form* to the 2-covariant skew-symmetric tensor field F defined by $F(X, Y) = g(X, JY)$.

An almost para-Hermitian manifold (M, g, J) such that $dF = 0$ shall be called an *almost para-Kähler manifold.*

Two almost para-Hermitian manifolds (M, g, J) and (M', g', J') are said to be *paraholomorphically isometric* if there exists an isometry $f \colon M \longrightarrow M'$ such that $f_* \circ J = J' \circ f_*$.

Proposition 4.2. ([50, 127]) *Let (M, g, J) be an almost para-Hermitian manifold, and let ∇ denote the Levi-Civita connection associated to g. Then, with the above notations, the following relation holds*

$$2g((\nabla_X J)Y, Z) + 3dF(X, Y, Z) + 3dF(X, JY, JZ) + g(JX, N(Y, Z)) = 0.$$

Several authors have defined almost para-Hermitian manifolds, with this name or another, as neutral manifolds, bilagrangian manifolds, almost hyperbolic Hermite manifolds, hyperbolic almost Hermite manifolds, etc. It seems that the first author to use the name hyperbolic in this context is Prvanović, but already Crumeyrolle used the term *hyperbolic complex structure* in [55]. Some results on the existence of almost para-Hermitian manifolds can be found in [109].

Definition 4.3. Let (M, g, J) be an almost para-Hermitian manifold, and let ∇ be the Levi-Civita connection of g. The curvature operator $R(X, Y)$ is defined by $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, and the Riemann-Christoffel tensor field is given by $R(X, Y, Z, W) = g(R(X, Y)Z, W)$. We shall denote also by R the value of R at a generic point $x \in M$. Then, if $X, Y \in T_xM$, we write $K'(X, Y) = R(X, Y, X, Y)$. A subspace $E \subset T_xM$ is said to be non-degenerate if $g|_E$ is non-degenerate. If $\{X, Y\}$ is a basis of a plane $E \subset T_xM$, then E is non-degenerate if and only if $g(X, X)g(Y, Y) - g(X, Y)^2 \neq 0$. For any non-degenerate plane $E \subset T_xM$ we define the sectional curvature as

$$K(X, Y) = \frac{K'(X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2},$$

where $\{X, Y\}$ is any basis of E; $K(X, Y)$ only depends on E. Since $g(JX, Y) + g(X, JY) = 0$, we have $g(X, JX) = 0$. If $X, JX \in T_xM$ are linearly independent, they determine a plane of T_xM, which we call the *paraholomorphic section defined by* X. If the sectional curvature is defined, that is, if X is not isotropic, we write $H'(X) = K'(X, JX)$, $H(X) = K(X, JX)$ and say that $H(X)$ is the *paraholomorphic sectional curvature defined by* X.

Paraholomorphic sectional curvature is introduced and studied in [80]. The authors obtain several results on such a curvature in the case of para-Kähler space forms (see Section 8).

4.2 Examples of almost para-Hermitian manifolds

(1) ([24]) Any parallelizable even-dimensional manifold and, in particular, any even-dimensional Lie group, can be endowed with an almost para-Hermitian structure.

(2) ([24, 82]) The cotangent bundle of any manifold admits an almost para-Hermitian structure. One way to give such a structure is the following: Let M be an n-dimensional differentiable manifold, endowed with a symmetric linear connection ∇ with coefficients Γ^i_{jk}. Then, one can define the Riemann extension on the cotangent bundle T^*M as the pseudo-Riemannian metric G on the total space of T^*M locally given by

$$G = \begin{pmatrix} -2p_k\Gamma^k_{ij} & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix},$$

with regard to the local coordinates (x_i, p_i) on T^*M. On the other hand, let J be the almost product structure on T^*M which has as vertical distribution the vertical subbundle of TT^*M, and as horizontal subbundle the horizontal distribution determined by the connection ∇. We have the following result:

Theorem 4.4. ([24]) *With the above notations, (G, J) is an almost para-Kähler structure on the total space of the cotangent bundle T^*M, whose fundamental 2-form F satisfies $F = d\theta$, where θ denotes the Liouville form, and thus coincides with the canonical symplectic structure on T^*M. Moreover, if ∇ has vanishing curvature, then the structure (G, J) is para-Kähler.*

(3) ([25]) Let $S^{2n-1}_n(r) = \{x \in \mathbb{R}^{2n}_n : \langle x, x\rangle = r^2\}$ be the pseudosphere of radius $r \geq 0$, dimension $2n-1$ and index n in $\mathbb{R}^{2n}_n$, and let $H^{2n-1}_n(r) = \{x \in \mathbb{R}^{2n}_{n+1} : \langle x, x\rangle = -r^2\}$ be the pseudohyperbolic space of radius $r \geq 0$, dimension $2n-1$ and index n in $\mathbb{R}^{2n}_{n+1}$. Then the product manifolds $M = S^{2n_1-1}_{n_1}(r_1) \times S^{2n_2-1}_{n_2}(r_2)$ and $M = H^{2n_1-1}_{n_1}(r_1) \times H^{2n_2-1}_{n_2}(r_2)$, $r_1, r_2 \geq 0$, $n_1, n_2 \in \{1, 2, ...\}$, admit a family of almost para-Hermitian structures. By analogy with Hopf's and Calabi-Eckmann's manifolds, Bejan calls *hyperbolic Hopf manifolds* and *hyperbolic Calabi-Eckmann manifolds* to the above product manifold M when either $n_1 = 1$ and $n_2 \in \{2, 3, \ldots\}$ or $n_1, n_2 \in \{2, 3, \ldots\}$, respectively.

(4) A lot of examples of almost para-Hermitian manifolds is given by Bejan in [25, refs. 12, 18, 25], some of which are included in the present survey. Those given in [21] are almost para-Hermitian structures on the tangent bundle TM of a manifold M, associated to vertical, complete and horizontal lifts [260] of tensor fields on M to TM.

(5) ([38]) Let TM be the tangent bundle of an n-dimensional manifold M. Let the total space of TM be endowed with a nonlinear connection D, and denote by X^v and X^h the vertical and the horizontal lift of a vector field X on M. Putting $J(X^v) = X^h$ and $J(X^h) = X^v$ we have an almost paracomplex structure on the manifold TM. Let g be a metric on the vertical bundle VTM. Then, writing

$$G(X^h, Y^h) = -G(X^v, Y^v) = g(X^v, Y^v), \qquad G(X^h, Y^v) = G(Y^v, X^h) = 0,$$

we obtain a metric G on TM of signature (n, n), and (G, J) is an almost para-Hermitian structure on TM.

(6) In [49, 50, 52] it is shown that if one considers a linear connection ∇ and a nonsingular either symmetric or skew-symmetric (0,2) tensor field g on a manifold M, then certain lifts of these objects to TM or to T^*M give a complete class of some important, tightly linked structures, including some remarkable almost paracomplex structures. In fact, consider on TM, besides the structures P and Q given by (3.3), the (0,2) tensor fields Ω, Θ , and K defined by

$$
\begin{aligned}
\Omega(X^h,Y^h)=\Omega(X^v,Y^v)=0,\quad & \Omega(X^h,Y^v)=\Omega(Y^v,X^h)=g(X,Y)^v\,;\\
\Theta(X^h,Y^h)=\Theta(X^v,Y^v)=0,\quad & \Theta(X^h,Y^v)=-\Theta(Y^v,X^h)=g(X,Y)^v\,;\\
K(X^h,Y^h)=-K(X^v,Y^v)=g(X,Y)^v,\quad & K(X^h,Y^v)=K(Y^v,X^h)=0\,.
\end{aligned}
\tag{4.2}
$$

If g is symmetric, then the structures (P,Ω) and (Q,K) are almost para-Hermitian structures with the same fundamental form, which is equal to Θ. If g is skew-symmetric, then the structures (P,Θ) and (Q,Θ) are also almost para-Hermitian, with fundamental 2-forms equal, respectively, to Ω and K.

Consider now on T^*M, besides the almost paracomplex structures P and Q given by (3.4), the (0,2) tensor fields Ω, Θ, H and K given by

$$
\begin{aligned}
\Omega(X^h,Y^h)=\Omega(\alpha^v,\beta^v)=0,\quad & \Omega(X^h,\beta^v)=\Omega(\beta^v,X^h)=\beta(X)^v\,,\\
\Theta(X^h,Y^h)=\Theta(\alpha^v,\beta^v)=0,\quad & \Theta(X^h,\beta^v)=-\Theta(\beta^v,X^h)=\beta(X)^v\,,\\
H(X^h,Y^h)=g(X,Y)^v,\quad & H(X^h,\beta^v)=H(\beta^v,X^h)=0\,,\\
H(\alpha^v,\beta^v)=g^{-1}(\alpha,\beta)^v,\quad & K(X^h,Y^h)=g(X,Y)^v\,,\\
K(X^h,\beta^v)=K(\beta^v,X^h)=0,\quad & K(\alpha^v,\beta^v)=-g^{-1}(\alpha,\beta)^v\,.
\end{aligned}
\tag{4.3}
$$

If g is symmetric, the structures (P,Ω) and (Q,K) are almost para-Hermitian structures with the same fundamental 2-form Θ. If g is skew-symmetric, the structures (P,Ω) and (Q,Ω) are almost para-Hermitian structures with fundamental 2-forms equal, respectively, to Θ and H. Notice that P, Ω and Θ depend only on the connection ∇ on M. Hence, any linear connection ∇ on a manifold M determines an almost para-Hermitian structure (P,Ω) on the cotangent bundle T^*M. If ∇ is symmetric, then the structure (P,Ω) is the one given in Example 2 above.

(7) ([82]) Let N_1 and N_2 be two arbitrary manifolds and g_1, g_2 two metrics defined on T^*N_1, T^*N_2, respectively, giving these a structure of almost para-Hermitian manifold. Write $T(T^*N_1)=P_1\oplus Q_1$ and $T(T^*N_2)=P_2\oplus Q_2$. Take as (M,g) the pseudo-Riemannian product of (T^*N_1,g_1) and (T^*N_2,g_2). Then $TM=P_1\oplus Q_1\oplus P_2\oplus Q_2$, and the two distributions $P=P_1\oplus Q_2$ and $Q=Q_1\oplus P_2$ on M define an almost product structure J on M. Now, since P_1 and Q_2 are both g-isotropic and $g(X,Y)=0$ for all $X\in P_1$, $Y\in Q_2$, we conclude that P is isotropic with respect to g. Similarly, Q is g-isotropic and hence (M,g,J) is an almost para-Hermitian manifold.

(8) Ianuş and Rosca [115] proved that, given a Riemannian structure g on a manifold M, and the almost paracomplex structure P on TM given in (3.4), being ∇ the Levi-Civita connection of g, then the complete lift g^c of g with respect to ∇ determines an almost para-Hermitian structure on TM.

(9) ([144]) The pseudosphere S^6_3 admits a structure of almost para-Hermitian manifold which is not integrable. Libermann obtains this structure by using Cayley's split octaves.

4.3 Representation-theoretical classification of almost para-Hermitian manifolds

As is well-known, in [107] almost Hermitian manifolds (M, g, J) are classified with respect to the decomposition in invariant and irreducible subspaces, under the action of the structural group $U(n)$, of the vector space of tensors satisfying the same symmetries as the covariant derivative ∇F of the fundamental 2-form F with respect to the Levi-Civita connection ∇ of the metric g. Thus we have an adequate framework for the several types of almost Hermitian manifolds, previously defined by a number of authors in terms of geometric properties which retain some portion of Kähler geometry. The previous method was used in [166] for Riemannian almost product manifolds, and in [104] for almost complex manifolds with a Norden metric. A classification of almost para-Hermitian manifolds is made in [20]. The author obtains 36 classes up to duality, and gives characterizations of some of the classes. A classification à la Gray-Hervella is given in [93], where 136 classes up to duality are obtained. We give here the table of primitive classes $\mathcal{W}_1, \ldots, \mathcal{W}_8$ obtained in [93]:

PRIMITIVE CLASSES OF ALMOST PARA-HERMITIAN MANIFOLDS OF DIMENSION ≥ 6								
	$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_3$	$\mathcal{W}_4$	$\mathcal{W}_5$	$\mathcal{W}_6$	$\mathcal{W}_7$	$\mathcal{W}_8$
$(\nabla_X F)(X, Y) = 0$	*				*			
$dF = 0$		*				*		
$\delta F = 0$			*				*	
$(\nabla_X F)(Y, Z) =$ $\{1/2(n-1)\}\{\delta F(Y)g(X, Z) - \delta F(Z)g(X, Y)$ $+\delta F(JY)g(X, JZ) - \delta F(JZ)g(X, JY)\}$				*				*
$\nabla_A B \in \mathcal{V}$			*	*	*	*	*	*
$\nabla_U B \in \mathcal{V}$	*	*	*	*	*	*		
$\nabla_A U \in \mathcal{H}$	*	*			*	*	*	*
$\nabla_U V \in \mathcal{H}$	*	*	*	*			*	*

Here, for an almost para-Hermitian manifold (M, g, J), ∇ denotes the Levi-Civita connection, F the fundamental 2-form, δ the codifferential, $\mathcal{V}$ (for *vertical*) the $(+1)$-eigendistribution associated to the eigenvalue $+1$ of J, $\mathcal{H}$ (for *horizontal*) the (-1)-eigendistribution corresponding to the eigenvalue -1 of J, A, B vector fields of $\mathcal{V}$ and U, V vector fields of $\mathcal{H}$.

The authors also give examples of the primitive classes, which are based on the general almost para-Hermitian structure on the tangent bundle given in [50].

Remark 4.5. The term "classification" is not the most correct. For this kind of results, it would be best to use another more appropriate term, as for instance: pre-classification, division, tabulation.

We now give a few commentaries on some classes related with the given classification:

The class of almost para-Hermitian manifolds is the most general one, corresponding to the total space $\mathcal{W} = \bigoplus_{i=1,\ldots,8} \mathcal{W}_i$.

The class of para-Kähler manifolds is the smallest class, corresponding to the trivial irreducible and invariant subspace $\{0\}$ of W.

The class of almost para-Kähler manifolds corresponds to the subspace $\mathcal{W}_2 \oplus \mathcal{W}_6$.

A class containing the locally conformal para-Kähler manifolds is the class $\mathcal{W}_4 \oplus \mathcal{W}_8$. As in the Hermitian case [107], the class of locally conformal para-Kähler manifolds does not coincide with $\mathcal{W}_4 \oplus \mathcal{W}_8$ (see [93]). The situation on this class is explained in Section 4.4.

The class of nearly para-Kähler manifolds, characterized by the condition $(\nabla_X J)X = 0$ with the usual notations. An example is Libermann's quadric S_3^6 with the structure given in [144] (see [24]).

Conversely, there exist many definitions of particular cases of almost para-Hermitian manifolds whose relation with the previous classification has not been studied. Among them, we recall here the manifolds defined and studied in [247]:

Definition 4.6. An almost para-Hermitian manifold (M, g, J) is said to be an *almost hyperbolic O^*-manifold* if $(\nabla_X F)Y + (\nabla_{JX} F)JY = 0, \quad X, Y \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection of g, and F is the fundamental 2-form.

4.4 Locally conformal para-Kähler manifolds

Locally conformal para-Kähler manifolds are the paracomplex analogs of locally conformal Kähler manifolds, introduced in [248].

Definition 4.7. Let (M, g_1, J_1) and (M, g_2, J_2) be two almost para-Hermitian manifolds. They are said to be locally conformally related if $J_1 = J_2$ and for every $x \in M$ there exists an open neighbourhood U of x such that g_1 and g_2 are conformally related in U.

Given a $2n$-dimensional almost para-Hermitian manifold (M, g, J), its *Lee form* θ is the 1-form on M defined by $\theta(X) = -(1/(n-1))\delta F(JX)$.

Proposition 4.8. ([93]) *An almost para-Hermitian manifold (M, g, J) is locally conformally related to a para-Kähler manifold (M, g_0, J) if and only if $N = 0$, $dF + \theta \wedge F = 0$ and $d\theta = 0$, where N denotes the Nijenhuis tensor of J and θ the Lee form of (M, g, J).*

Remark 4.9. Let Ξ be the tensor field on the almost para-Hermitian manifold (M, g, J) given by $g(\Xi(X,Y), Z) = (dF + \theta \wedge F)(X, Y, Z)$. The three tensor fields N, Ξ and $d\theta$ are global conformal invariants of the structure. Moreover, it should be noted that if one defines a tensor field Υ on M by the formula

$$
\begin{aligned}
g(\Upsilon(X,Y), Z) = (\nabla_X F)(Y, Z) - \frac{1}{2(n-1)}\{&\delta F(Y) g(X, Z) - \delta F(Z) g(X, Y) \\
&+ \delta F(JY) g(X, JZ) - \delta F(JZ) g(X, JY)\}, \quad X, Y, Z \in \mathfrak{X}(M),
\end{aligned}
$$

then the following can be proved:

Proposition 4.10. ([93])

(1) *The tensor field Υ is a global conformal invariant.*

(2) $\Upsilon = 0$ *if and only if* $N = 0$ *and* $dF + \theta \wedge F = 0$, *or equivalently, if and only if* $\nabla_A B \in \mathcal{V}$, $\nabla_U V \in \mathcal{H}$, $A, B \in \mathcal{V}$, $U, V \in \mathcal{H}$ *and* $dF + \theta \wedge F = 0$.

From Propositions 4.8 and 4.10 it follows that any para-Hermitian manifold locally conformally equivalent to a para-Kähler manifold is a manifold which belongs to the class $\mathcal{W}_4 \oplus \mathcal{W}_8$. We also note that *the pair $(\Upsilon, d\theta)$ is the analog of the Weyl conformal tensor of Riemannian geometry.* An example of an almost para-Hermitian manifold with $\Upsilon = 0$ but which is not locally conformally equivalent to a para-Kähler manifold is given in [93].

Example 4.11. ([25]) The product manifold $M = S^1 \times H^{2n-1}_{n-1}(r)$, $r > 0$, $n \in \{2, 3, \ldots\}$, with the structure given in that reference, is a locally conformal para-Kähler manifold which cannot be globally conformal para-Kähler.

Remark 4.12. Other classifications of almost para-Hermitian manifolds.
There are other classifications of almost para-Hermitian manifolds:

(1) A classification related to the parallelism properties of the structural distributions is given in [81].

(2) In [183] (and independently in [104]) the vector space $K(M)$ of (0,4) tensors which satisfy the same symmetries as the Riemann-Christoffel curvature tensor R and also the new symmetry satisfied by para-Kähler manifolds $R(X, Y, JZ, W) + R(X, Y, Z, JW) = 0$ is considered. With regard to the metric induced by g in this vector space, it decomposes in 3 invariant orthogonal subspaces: $K(M) = K_1(M) \oplus K_2(M) \oplus K_B(M)$, where

$$\begin{aligned} K_1(M) &= \{R \in K(M) : R \text{ is of constant paraholomorphic curvature}\}, \\ K_2(M) &= \{R \in K(M) : R \text{ is of null scalar curvature}\}, \\ K_B(M) &= \{R \in K(M) : R \text{ has null Ricci tensor and null scalar curvature}\}. \end{aligned}$$

In [183] the expression of the components of a vector $R \in K(M)$ relative to this decomposition is given: $R = R_1 + R_2 + R_B$, and R_B is called the *Bochner tensor* associated to R.

Corollary 4.13. ([104]) *A para-Kähler manifold M with dimension $2n \geq 4$ has constant paraholomorphic sectional curvature if and only if M is Einstein and $R_B(M) = 0$.*

4.5 Compatible linear connections

The description of the family of connections compatible with an almost paracomplex structure or with an almost para-Hermitian structure follows immediately from the expression obtained in [47] for the general case of Banach vector bundles: Let J be a nonsingular (1,1) tensor field and g a nonsingular (0,2) tensor field on a manifold M, and consider the relations

$$(a) \quad J^2 = \varepsilon_1 I, \qquad (b) \quad {}^t g = \varepsilon_2\, g, \qquad (c) \quad {}^t J \circ g = \varepsilon_3\, g \circ J\,,$$

where I is the Kronecker tensor on M and ε_1, ε_2, $\varepsilon_3 = \pm 1$. Then:

(i) If J satisfies the condition (a), the family of connections parallelizing J is given by

$$\nabla = \Phi_J(\overset{\circ}{\nabla}) + \Psi_J(\sigma)\,; \tag{4.4}$$

(ii) If g satisfies the condition (b), the family of connections parallelizing g is given by

$$\nabla = \Phi_g(\overset{\circ}{\nabla}) + \Psi_g(\sigma)\,; \tag{4.5}$$

(iii) If J and g satisfy (a), (b) and (c), the family of connections parallelizing J and g is given by

$$\nabla = \Phi_J \circ \Phi_g(\overset{\circ}{\nabla}) + \Psi_J \circ \Psi_g(\sigma)\,; \tag{4.6}$$

where $\overset{\circ}{\nabla}$ is a fixed arbitrary linear connection, σ any (1,2) tensor field on M, and Φ_J, Φ_g, Ψ_J and Ψ_g are given by

$$\Phi_J(\overset{\circ}{\nabla})_X = \overset{\circ}{\nabla}_X + \frac{1}{2}J^{-1}\circ \overset{\circ}{\nabla}_X J, \quad \Psi_J(\sigma)_X = \frac{1}{2}(\sigma_X + J^{-1}\circ\sigma_X \circ J)\,,$$
$$\Phi_g(\overset{\circ}{\nabla})_X = \overset{\circ}{\nabla}_X + \frac{1}{2}g^{-1}\circ \overset{\circ}{\nabla}_X g, \quad \Psi_g(\sigma)_X = \frac{1}{2}(\sigma_X + g^{-1}\circ\sigma_X \circ g)\,,$$

for every $X \in \mathfrak{X}(M)$.

Formula (4.4) gives, for $\varepsilon_1 = 1$, the family of connections compatible with an almost para-complex structure. Formula (4.5) furnishes, for $\varepsilon_2 = 1$, the family of connections compatible with a pseudo-Riemannian structure; and, for $\varepsilon_2 = -1$, the family of connections compatible with an almost symplectic structure. Formula (4.6) gives, for $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$, the family of connections compatible with an almost para-Hermitian structure.

The family of linear connections parallelizing an almost para-Hermitian structure can be also obtained immediately from the general expression given in [196]. In [156], from the quoted general results in [47], the family of connections compatible with an almost para-Hermitian structure is also obtained. On the other hand, some partially adapted connections are obtained in [197].

Given an almost para-Hermitian manifold (M,g,J), there exists a unique linear connection ∇, with torsion T, satisfying the conditions:

$$\nabla g = 0, \quad \nabla J = 0, \quad T(PX,QY) = 0, \qquad X,Y \in \mathfrak{X}(M)\,,$$

where

$$P = \frac{1+J}{2}, \quad Q = \frac{1-J}{2}\,,$$

are the projectors of J. This connection is called the *canonical connection.*

Let $T^+(M)$ and $T^-(M)$ denote the eigenbundles of J, corresponding to the eigenvalues $+1$ and -1 respectively. Then, we have for ∇:

$$\nabla_{X^+}X^- = Q[X^+,X^-], \quad \nabla_{X^-}X^+ = P[X^-,X^+],$$
$$g(\nabla_{X^+}Y^+,Z^-) = X^+g(Y^+,Z^-) - g([X^+,Z^-],Y^+),$$
$$g(\nabla_{X^-}Y^-,Z^+) = X^-g(Y^-,Z^+) - g([X^-,Z^+],Y^-),$$
$$X^+,Y^+,Z^+ \in T^+(M),\ X^-,Y^-,Z^- \in T^-(M).$$

Let us consider the tensor

$$\Psi(X,Y,Z) = g(X,T(Y,Z)), \qquad X,Y,Z \in \mathfrak{X}(M)\,,$$

where T is the torsion tensor of the canonical connection of the almost para-Hermitian structure (g,J) on the manifold M. One can prove that Ψ has the same symmetries as the tensor Φ considered in [20] and [93] to obtain the classifications of almost para-Hermitian manifolds. Hence, one can obtain a classification of almost para-Hermitian structures given in terms of Ψ which coincides with the classification based on Φ but with different characterizations.

4.6 Reflectors

Reflectors were introduced in the study of neutral surfaces in 4-dimensional neutral pseudo-Riemannian manifolds [117], and they are the neutral space analogs of the twistor spaces of Riemannian geometry. We recall here some basic facts and definitions from [117], where many other results on reflectors are obtained.

Let V be a real vector space of even dimension endowed with a neutral metric g and a paracomplex structure J such that (V, g, J) is para-Hermitian. If (V, g) is oriented, then the paracomplex structures J divide into two disjoint categories, called positively or negatively oriented, depending on whether the adapted null frames with respect to which the structure is defined are positively or negatively oriented.

Let now $G_{1,1}(2,2)$ be the Grassmannian of oriented neutral planes in $\mathbb{R}^4_2$. The non-semisimple group $SO(2,2)$ acts transitively on $G_{1,1}(2,2)$, which becomes a homogeneous space $G_{1,1}(2,2) \approx SO(2,2)/SO(1,1) \times SO(1,1)$. It can be proved that $G_{1,1}(2,2)$ admits an $SO(2,2)$-invariant para-Hermitian structure arising from a $SO(2,2)$-invariant metric related to the Maurer-Cartan form of $SO(2,2)$. Consider the groups $B_\pm$ defined as follows: B_+ is the group $B(2)$, particular case for $n = 2$ of the usual almost para-Hermitian structural group

$$B(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \in GL(2n, \mathbb{R}) : A \in GL(n, \mathbb{R}) \right\},$$

and B_- is the conjugate group of B_+ in $O(2,2)$ defined by

$$B_- = \{A \in SO(2,2) : AI_- = I_-A, \quad I_- = \mathrm{diag}(-1,1,1,-1)\}.$$

Let $S_\pm = SO(2,2)/B_\pm$, and let $\pi_\pm\colon SO(2,2) \longrightarrow SO(2,2)/B_\pm = S_\pm$ denote the projections. S_+ and S_- are endowed with certain $SO(2,2)$-invariant neutral metrics g_+ and g_-, respectively, each with Gauss curvature equal to 2. There is moreover an $SO(2,2)$-equivariant isometry

$$\psi = (\psi_+, \psi_-)\colon G_{1,1}(2,2) \longrightarrow S_+ \times S_- . \tag{4.7}$$

Let now $f\colon S \longrightarrow \mathbb{R}^4_2$ be an isometric immersion of an oriented neutral surface, and let its Gauss map be $\gamma_f\colon S \longrightarrow G_{1,1}(2,2)$. From the above splitting of the Grassmannian the Gauss map factors into $\gamma_f = (\varphi_+, \varphi_-)$, where $\varphi_\pm = \psi_\pm \circ \gamma_f$. The maps $\varphi_\pm$ are called the *reflector maps* of f, and are the analogs of twistor maps for surfaces in $\mathbb{R}^4$.

Let (M, g) be a 4-dimensional neutral pseudo-Riemannian manifold, that is, such that the pseudo-Riemannian metric g has signature $(++--)$, and denote by $SO(M)$ the bundle of orthonormal frames on M. If (M, g) has holonomy, there is no unambiguous way to transport tangent 2-planes parallelly to a fixed origin of M. In order to generalize the Gauss map for an oriented neutral surface $f\colon S \longrightarrow (M, g)$, one can consider the Grassmann bundle $G_{1,1}(M)$ of oriented neutral planes at the points of M. Generalizing the previous situation, one can consider sections of the Grassmann bundle

$$G_{1,1}(M) = \{(p, P) : P \text{ is an oriented neutral plane in } T_p(M)\}.$$

From the standard action of $SO(2,2)$ on $G_{1,1}(2,2)$, we have

$$G_{1,1}(M) \approx SO(M) \times_{SO(2,2)} G_{1,1}(2,2) = SO(M)/SO(1,1) \times SO(1,1).$$

The Gauss lift of f is given by the map $\gamma_f\colon S \longrightarrow G_{1,1}(M)$, where $\gamma_f(p) = (f(p),\, f_*T_pM)$.

The splitting of $G_{1,1}(2,2)$ given in (4.7) leads us to consider the *reflector bundles*

$$r_\pm\colon Z_\pm \longrightarrow M, \tag{4.8}$$

defined by

$$Z_\pm = \{(p, J) : J \text{ is an almost para-Hermitian structure on } (T_pM, g|_p) \text{ of } \pm \text{ orientation}\}$$
$$\approx \quad SO(M) \times_{SO(2,2)} S_\pm \quad \approx \quad SO(M)/B_\pm\,.$$

The projection map is defined by $r_\pm(p, J) = p$.

Let $\sigma_\pm$ denote the natural projection $\sigma_\pm \colon SO(M) \longrightarrow Z_\pm = SO(M)/B_\pm$. There are natural maps $\psi_\pm \colon G_{1,1}(M) \longrightarrow Z_\pm$, where $\psi_\pm(p, P) = (p, J_\pm)$. The reflector lifts of f are $\varphi_\pm = \psi_\pm \circ \gamma_f$, so that $\varphi_\pm \colon M \longrightarrow Z_\pm$ are sections of (4.8) in the sense that $r_\pm \circ \varphi_\pm = f$, i.e., they are sections of $f^{-1}Z_\pm \to M$.

4.7 Submanifolds of almost para-Hermitian manifolds

The hypersurfaces of almost para-Hermitian manifolds are considered in [178], where it is proved that they admit a hyperbolic contact structure. A condition for a hypersurface of this type with vanishing curvature tensor to be conformally flat is also given. In [66], hypersurfaces in almost para-Hermitian manifolds are studied.

Three types of submanifolds of an almost para-Hermitian manifold *with the Killing property* in the sense of [57] are studied in [12], and it is proved that those submanifolds are minimal.

The immersions of a hyperbolic surface *with the geodesic property* in the sense of [203] into a neutral pseudo-Riemannian manifold are studied in [13].

The study of degenerate hypersurfaces of almost para-Hermitian manifolds is initiated in [28]. The authors consider the general case, and then some important particular cases, as totally geodesic degenerate hypersurfaces, totally umbilical degenerate hypersurfaces and minimal degenerate hypersurfaces.

A detailed study of degenerate hypersurfaces of almost para-Hermitian manifolds, including the geometric structures induced on such submanifolds and the totally umbilical case is made in [27].

The classification of degenerate and non-degenerate submanifolds, and of CR-submanifolds (see below) of a 4-dimensional almost para-Hermitian manifold is given in [77]. These classifications rely in the fact observed by the authors, that the geometry of a submanifold (M, g, J) of an almost para-Hermitian manifold $(\bar{M}, \bar{g}, \bar{J})$ of any dimension is determined, to a high degree, by $\dim(T_pM \cap T_pM^\perp)$ and $\bar{J}(T_pM)$. Their technique can be extended to higher dimensions.

General Cauchy-Riemann submanifolds (CR-submanifolds), and the special case of totally umbilical Cauchy-Riemann submanifolds are studied in [22, 25, 75, 76]:

Definition 4.14. Let (N, g, J) be an almost para-Hermitian manifold. A pseudo-Riemannian submanifold M of N is said to be a *CR-submanifold* if the following conditions are satisfied:
(1) The metric $g_M = g|_M$ is of constant signature and rank.
(2) There exist two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M satisfying:
  (i) $\mathcal{D}$ is invariant, i.e., $J\mathcal{D}_p = \mathcal{D}_p$ for all $p \in M$.
  (ii) $\mathcal{D}^\perp$ is anti-invariant, i.e., $J\mathcal{D}^\perp_p \subset (T_pM)^\perp$ for all $p \in M$.
  (iii) $T_pM = \mathcal{D}_p \oplus \mathcal{D}^\perp_p$, for all $p \in M$, and $\mathcal{D}_p$, $\mathcal{D}^\perp_p$ are mutually orthogonal.
Moreover, we have the following types of CR-submanifolds:
(a) *Invariant*, if $\mathcal{D}^\perp = \{0\}$.
(b) *Anti-invariant*, if $\mathcal{D} = \{0\}$.
(c) *Proper CR-submanifold*, if $\mathcal{D} \neq \{0\}$ and $\mathcal{D}^\perp \neq \{0\}$.
(d) *Generic*, if $\dim \mathcal{D}^\perp_p = \dim(T_pM)^\perp \neq 0, \quad p \in M$.

4.8 Examples of CR-submanifolds

(1) ([8]) Every *principal co-isotropic submanifold of index* 1 of a para-Kähler manifold *with the Poisson property* M is a proper CR-submanifold of M such that both $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are involutive.

(2) ([22]) Consider the 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ endowed with the Riemannian metric g which is the product of the standard metrics on the factors S^1 which makes the vector fields $\{X_i\}$, $i = 1, 2, 3$, representing the parallelism, an orthonormal global frame such that X_3 is normal to T^2 and X_1 tangent to $S^1 \times \{\theta\} \times \{\theta\}$, where $S^1 \times \{\theta\} \times \{\theta\} \hookrightarrow T^2 \times \{\theta\} \hookrightarrow T^3$. Let $M = T^3 \times T^3$ and define on M the pseudo-Riemannian structure $\tilde{g}$ and the almost product structure J given at each point (p, q) by

$$\tilde{g} = \begin{pmatrix} -g & 0 \\ 0 & g \end{pmatrix} \quad \text{and} \quad J_{(p,q)} = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix},$$

with regard to $\{(X_i, 0),\ i = 1, 2, 3; \quad (0, X_k),\ k = 1, 2, 3\}$. Then, the manifold $(M = T^3 \times T^3, \tilde{g}, J)$ is an almost para-Hermitian manifold which contains $T^2 \times S^1$ as a non-generic proper CR-submanifold.

(3) ([22]) Let (M_i, g_i, J_i), $i = 1, 2$, be two almost para-Hermitian manifolds with $\dim M_i > 2$. Then the product manifold $M_1 \times M_2$, endowed with

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix},$$

is an almost para-Hermitian manifold and any pseudo-Riemannian hypersurface of M_1 is a proper non-generic CR-submanifold.

(4) Some basic facts on CR-submanifolds of almost para-Hermitian manifolds are stated in ([75, 76]). The authors give examples of CR-submanifolds of paracomplex projective spaces, prove that degenerate hypersurfaces are CR-submanifolds if and only if they are invariant and give a classification of submanifolds and CR-submanifolds of a 4-dimensional almost para-Hermitian manifold.

R. Rosca and other authors have also considered CR-submanifolds in a para-Kählerian manifold [208, 210, 212].

Many properties of the almost para-Hermitian manifolds *structured by an $\mathfrak{A}$-conformal connection* are studied in [14]. Similarly, in [15], several properties of the almost para-Hermitian manifolds *with geodesic connection* are proved.

4.9 Homogeneous almost para-Hermitian structures

The classical characterization by Ambrose and Singer [11] of homogeneous Riemannian manifolds in terms of a (1,2) tensor field S on the manifold, which is an extension of Cartan's characterization [40] of Riemannian symmetric manifolds (for which one has $S = 0$) is extended to pseudo-Riemannian manifolds of arbitrary signature in [101]. The authors give the following:

Definition 4.15. A *homogeneous almost para-Hermitian structure* on the almost para-Hermitian manifold (M, g, J) is a (1,2) tensor field S on M such that the connection $\tilde{\nabla} = \nabla - S$, where ∇ denotes the Levi-Civita connection of the metric g, parallelizes the metric g, its curvature R and the tensor fields J and S.

Theorem 4.16. ([101]) *Let (M, g, J) be a connected, simply connected and complete almost para-Hermitian manifold of dimension $2n$. Then (M, g, J) admits a homogeneous almost para-Hermitian structure if and only if it is a reductive homogeneous almost para-Hermitian manifold $(M = G/H, g, J)$.*

Notice that in the Riemannian case a homogeneous manifold is always complete and reductive.
A classification of homogeneous almost para-Hermitian structures is given in [102], where 175 classes are obtained.

Example 4.17. ([101]) Libermann's quadric [144] S_3^6 can be viewed either as the pseudo-Riemannian manifold of constant curvature $O(3, 4)/O(3, 3)$, which is pseudo-Riemannian symmetric, with corresponding homogeneous pseudo-Riemannian structure $S = 0$, or well as the homogeneous space $G_2'/SL(3, \mathbb{R})$, where G_2' denotes the exceptional simple Lie group which is the second real form with fundamental group $\mathbb{Z}_2$ of the complex group of which the usual group G_2 is the compact real form. The isotropy group is the special paraunitary group isomorphic to the real special linear group of order 3. The homogeneous space $G_2'/SL(3, \mathbb{R})$ admits an almost para-Hermitian structure (g, J) which is not para-Kähler, but nearly para-Kähler [24], and it is a reductive almost para-Hermitian manifold with homogeneous almost para-Hermitian structure $S = -\frac{1}{2}\, J \circ (\nabla J)$, where ∇ denotes the Levi-Civita connection of g.

4.10 Examples of para-Hermitian manifolds

(1) ([49]) Para-Hermitian manifolds of a certain type arise as a particular case from a more general situation: that given on the tangent bundle TM of a manifold M endowed with a linear connection and a tensor field of type (1,1) or (0,2). Specific examples of para-Hermitian structures are given by Cruceanu's structures considered in Section 4.2, that is, (P, Ω), (Q, K), (P, Θ) and (Q, Θ) on TM or (P, Ω), (Q, K) and (Q, Ω) on T^*M, if the almost paracomplex structures P and Q are integrable.

On the other hand, the behaviour of para-Hermitian structures on a manifold with regard to the action of the multiplicative group of nonsingular tensor fields of type (1,1) on the tensor algebra, on its algebra of derivations and on the affine module of linear connections arises as a particular case in [51].

(2) ([117]) The 2-dimensional case has some interesting features: Let (M, g) be an oriented 2-dimensional neutral manifold. The metric g and the orientation of M induce a unique almost para-Hermitian structure J, which is automatically integrable since M is 2-dimensional, and it can thus be proved that there exists a kind of "*isothermal coordinates*" x, y, such that the metric can be locally written as $g = 2\rho\, dx\, dy$, where ρ is a positive C^∞ function.

(3) ([117]) Let $G_{1,1}(2, 2) \approx SO(2, 2)/SO(1, 1) \times SO(1, 1)$ be the Grassmannian of oriented neutral planes in $\mathbb{R}_2^4$. Then $G_{1,1}(2, 2)$ admits a para-Hermitian structure obtained from an $SO(2, 2)$-invariant metric related to the Maurer-Cartan form of $SO(2, 2)$.

(4) Some examples of (almost) para-Hermitian manifolds which are hypersurfaces of hyperbolic almost paracontact manifolds are given in [240].

Remark 4.18. Subtypes of almost para-Hermitian manifolds

(a) The *almost para-Hermitian manifolds with exterior recurrent line element splitting* are studied in [62].

(b) The almost para-Hermitian manifolds *structured by a parallel conformal connection* are considered in [63].

5 Para-Kähler manifolds

5.1 Definitions and first properties

Definition 5.1. A para-Hermitian manifold (M, g, J) is said to be a *para-Kähler manifold* if $dF = 0$. From Theorem 4.2, we deduce that, equivalently, a *para-Kähler manifold* is an almost para-Hermitian manifold such that $\nabla J = 0$, where ∇ denotes the Levi-Civita connection of g. We can also define, from [172] or from the classification in Subsection 4.3, a *para-Kähler manifold* as a pseudo-Riemannian manifold of dimension $2n$ endowed with two n-dimensional totally isotropic and parallel distributions $\mathcal{V}$ and $\mathcal{H}$ such $\mathcal{V} \cap \mathcal{H} = \{0\}$.

If we consider a connected (almost) para-Hermitian or para-Kähler manifold (M, g, J) and denote by $I(M, g)$ the isometry group of M with respect to g, then the *automorphism group* of (M, g, J) is defined as

$$\mathrm{Aut}(M, g, J) = \mathrm{Aut}(M, g) \cap \mathrm{Aut}(M, J),$$

which is a closed subgroup of $\mathrm{Aut}(M, g)$, and consequently a Lie transformation group of M. If $\mathrm{Aut}(M, g, J)$ acts transitively on M, then (M, g, J) is called a *homogeneous (almost) para-Hermitian* or *para-Kähler manifold*, as long as it is (almost) para-Hermitian or para-Kähler. Notice that a homogeneous almost para-Kähler manifold is a homogeneous symplectic manifold with respect to the fundamental form F and $\mathrm{Aut}(M, g, J)$.

Canonical forms for the metrics of all non-decomposable locally symmetric para-Kähler spaces were obtained in [181]. We recall here the following:

Theorem 5.2. ([181]) *The metric of a non-decomposable locally symmetric* 4*-dimensional para-Kähler space can be expressed in one of the following ways:*

(1) $g = \lambda(z^2 dx^2 + 2zt\, dx\, dy + t^2 dy^2) + 2dx\, dz + 2dy\, dt.$

(2) $g = \lambda z^2 dy^2 + 2dx\, dz + 2dy\, dt.$

(3) $g = z^2 dx\, dy + (\lambda z^2 + \frac{1}{2} zt) dy^2 + 2dx\, dz + 2dy\, dt,$

where λ denotes a constant.

In cases (1) *and* (2)*, the spaces are Einstein spaces, the scalar curvature being non-zero in case* (1) *and zero in case* (2)*. Moreover, those metrics are canonical forms for the metrics of all symmetric harmonic* 4*-spaces which do not have constant curvature. In case* (3) *the spaces are not Einstein spaces and so they are not harmonic.*

The Riemann-Christoffel curvature tensor of para-Kähler manifolds has the following new symmetry:

Proposition 5.3. *Let (M, g, J) be a para-Kähler manifold. Then its Riemann-Christoffel curvature tensor R satisfies the usual symmetries and also the symmetry*

$$R(X, Y, JZ, W) + R(X, Y, Z, JW) = 0, \quad X, Y, Z, W \in \mathfrak{X}(M).$$

A detailed and thorough study of curvature and curvature functions on para-Kähler manifolds is given by Vázquez Lorenzo in [250]. He introduces (among other concepts) anti-paraholomorphic sectional curvature and from this obtains many results as with paraholomorphic sectional curvature.

5.2 Examples of para-Kähler manifolds

(1) ([144]) The para-Kähler structure on $\mathbb{R}^{2n}$ given by the pseudo-Euclidean inner product $\langle\, , \rangle$ and the almost product structure J_{can} defined by

$$\langle\, , \rangle = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}, \quad J_{\text{can}} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

where both matrices are taken with respect to the canonical basis of $\mathbb{R}^{2n}$, is called the *canonical para-Kähler structure on* $\mathbb{R}^{2n}$.

(2) A *bilagrangian symplectic manifold* is a C^∞ manifold endowed with a closed 2-form F, a pseudo-Riemannian metric g and a couple of supplementary integrable distributions $\mathcal{D}_+$, $\mathcal{D}_-$, which are isotropic with respect to the metric g, that is, $F|_{\mathcal{D}_+} = 0$ and $F|_{\mathcal{D}_-} = 0$.

Such a manifold is said to be a *parallel bilagrangian symplectic manifold* if $\nabla F = 0$, where ∇ denotes the Levi-Civita connection of g. The relation of the above definitions (see [248]) with the definitions of almost para-Kähler and Kähler manifolds is clear.

(3) ([119]) Para-Kähler manifolds naturally appear in the geometry of negatively curved manifolds. Suppose that N is a simply connected complete Riemannian manifold with sectional curvature $K \leq -1$. Then the unit tangent bundle $S(N)$ of N is fibred over the space M of geodesics of N so that each fibre is an orbit of the geodesic flow of N. The exterior derivative $d\theta$ of the canonical contact form of $S(N)$, which is invariant by the geodesic flow, is pushed forward to a symplectic form F of M by the fibre bundle $S(N) \to M$.

On the other hand, we have that for a closed Riemannian manifold N of negative curvature, the geodesic flow defined in the unit tangent bundle $S(N)$ is an Anosov flow. Then the splitting $S(N) = E^- \oplus E^0 \oplus E^+$, which are called the *Anosov splitting* associated with the geodesic flow ϕ_t of N, determines foliations $\mathcal{E}^-$ and $\mathcal{E}^+$ called the (strongly) stable and unstable foliations of ϕ_t.

In the case of complete simply connected Riemannian manifolds with sectional curvature ≤ -1, the stable and unstable foliations $\mathcal{E}^+$ and $\mathcal{E}^-$ of $S(N)$, descend to foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ of M, which are transverse Lagrangian foliations of (M, F). We thus have an almost para-Kähler manifold, namely $(M, F, \mathcal{F}^+, \mathcal{F}^-)$, associated to the negatively curved manifold N. Moreover, in the case when N is the universal covering of a closed Riemannian manifold whose Anosov splitting is C^∞, the Lagrangian foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ are smooth, and M is para-Kähler.

(4) ([50]) The almost para-Hermitian structure (G, J) on the tangent bundle TM of a Riemannian manifold (M, g) endowed with a linear connection ∇ is almost para-Kähler if and only if the pair (g, ∇) is cotorsionless; that is, if its cotorsion τ vanishes everywhere. We recall that the *cotorsion* is defined in [49] as

$$\tau(X, Y, Z) = (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) + g(T(X, Y), Z), \quad X, Y, Z \in \mathfrak{X}(M),$$

T being the torsion of ∇. The structure (G, J) is para-Kähler if and only if (g, ∇) has vanishing cotorsion and curvature. Notice that G above coincides with Ω in the relations (3.3) and that J

above coincides with P in the relations (4.2). As for the cotorsion, it coincides with the exterior covariant differential of g given by Dg on page 352.

(5) ([82]) The cotangent bundle T^*M of a flat Riemannian manifold (M, g) admits a para-Kähler structure.

(6) ([82]) Let $M = \mathbb{R}^2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$ as the usual basis. Define a metric g on M such that for every point $(x, y) \in M$, g is given by the matrix

$$\begin{pmatrix} \frac{1}{2}e^{x+y} & 0 \\ 0 & -\frac{1}{2}e^{x-y} \end{pmatrix}.$$

Therefore (M, g) is not flat. If one takes $\mathcal{D}_+$, $\mathcal{D}_-$ to be the 1-distributions induced by $e_1 + e_2$ and $e_1 - e_2$ respectively, then $\mathcal{D}_+$ and $\mathcal{D}_-$ are isotropic. If J is the almost product structure given by $\mathcal{D}_+$ and $\mathcal{D}_-$ then (M, g, J) is a para-Kähler manifold which is not flat.

(7) ([82]) Every 2-dimensional almost para-Hermitian manifold is para-Kähler.

(8) ([33]) Let (M, g) be a para-co-Kähler manifold, σ an almost cosymplectic structure on M, and $H(M)$ the set of *horizontal vector subspaces* of σ. Then the hypersurfaces tangent to $H(M)$ are para-Kähler hyperspheres. See also [105] for the para-Kähler hyperspheres.

5.3 Subtypes of para-Kähler manifolds

Some results on *conformally flat para-Kähler manifolds* are obtained in [172], among which we remark the following:

Theorem 5.4. *Let M be a conformally flat para-Kähler manifold. Then:*

(1) *M is locally flat if* $\dim M \geq 6$.

(2) *M is locally symmetric and its scalar curvature vanishes identically if* $\dim M = 4$.

(3) *If M has* $\dim = 4$, *and the metric g of M is decomposable, then M is either:*

  (a) *locally flat,*

  (b) *locally a product $M_1 \times M_2$ of two 2-dimensional para-Kähler spaces M_1 and M_2, where M_1 is of constant Gauss curvature $K > 0$ and M_2 is of constant Gauss curvature $-K$.*

(4) *If M is 4-dimensional and the metric is not locally decomposable, then in appropriate coordinate systems, the para-Kähler structure of M can be given as in Theorem* 5.2 *with $\lambda = 0$.*

We also remark the plentiful results obtained by Rosca and his coworkers on para-Kähler manifolds endowed with supplementary structures.

Several tensors fields on almost para-Hermitian, almost para-Kähler and para-Kähler manifolds are considered in [18], where, among other results, necessary and sufficient conditions for a manifold in one of these classes to belong to a more restricted class are obtained.

We also have the following particular cases of para-Kähler manifolds:

(1) *Para-Kähler manifolds having the Poisson property* ([17]). Consider a para-Kähler manifold (M, g, J), and let $F = \sum_k \theta^k \wedge \theta^{n+k}$, $k = 1, \ldots, n$, be the canonical symplectic 2-form.

The manifold (M, g, J) is said to have the Poisson property if for each $p \in M$, there exists a neighbourhood U such that the Poisson brackets $\{\theta^k, \theta^{n+k}\}_p$, $p \in U$, of all the pairs (θ^k, θ^{n+k}) with regard to the symplectic structure of M are zero.

Theorem 5.5. ([17]) *Any para-Kähler manifold M having the Poisson property has the divergence property; that is, each $p \in M$ has a neighbourhood with an adapted local frame whose vectors have vanishing divergence.*

(2) Other types of para-Kähler manifolds are considered:

(a) In [7, 16], where the *para-Kähler manifolds with the concircular property* are defined and studied.

(b) Many properties of the *para-Kähler manifolds with self-orthogonal connection* are proved in [36].

(c) In [79], where the *para-Kähler manifolds having the skew-symmetric Killing property* are introduced and studied.

5.4 Submanifolds of para-Kähler manifolds

(1) ([22]) The pseudosphere $S_n^{2n-1}(r)$ and also the pseudohyperbolic space $H_n^{2n-1}(r)$ of radius r are examples of totally umbilical CR-submanifolds of the para-Kähler manifold $(\mathbb{R}_n^{2n}, \langle\ ,\ \rangle, J_{\mathrm{can}})$.

(2) The *$CICR$-submanifolds* (i.e., co-isotropic Cauchy-Riemann) of para-Kähler manifolds *having the self-orthogonal Killing property* are studied with detail in [64].

Remark 5.6. Characteristic classes of para-Kähler manifolds. The explicit expression of the Pontrjagin forms of Bochner flat para-Kähler manifolds with constant scalar curvature has been obtained by Bejan in [23].

The fact that every almost para-Hermitian manifold admits an almost complex structure is proved in [82]. Thus, the usual obstructions for this structure in terms of characteristic classes can be useful for the almost para-Hermitian case.

6 Transformations

6.1 Paraholomorphic curvature tensors

Prvanović's paper [188] deserves a special mention, as it not only contains several interesting definitions and results on paracomplex and para-Hermitian geometries, but encouraged the research on the subject, since one can see many differences between complex and paracomplex geometry. Prvanović introduced, among other things, the paraholomorphic projective curvature tensor, and also gave the explicit expression of the curvature tensor for spaces with constant paraholomorphic sectional curvature. On the other hand, the paraholomorphically projective curvature tensor is defined in [239] and a proof of the fact that if a manifold is paraholomorphically projectively flat, then it has constant paraholomorphic sectional curvature, is given.

Definition 6.1. ([188, 198]) The *paraholomorphic projective curvature tensor* P of a para-Kähler manifold (M, g, J) with dimension $2n$ is defined as

$$\begin{aligned} P(X,Y)Z = R(X,Y)Z - \frac{1}{2(n+1)}\{&S(Y,Z)X - S(X,Z)Y \\ &+ S(Y,JZ)JX - S(X,JZ)JY - 2S(X,JY)JZ\}, \quad X,Y,Z \in \mathfrak{X}(M), \end{aligned}$$

where R and S denote respectively the curvature tensor and the Ricci tensor.

Theorem 6.2. *Let (M, g, J) be a para-Kähler manifold of dimension > 2. Then it is paraholomorphically projectively flat if and only if $P \equiv 0$ or, equivalently, if (M, g, J) has constant paraholomorphic sectional curvature.*

Bejan [25], by using Prvanović's paraholomorphic curvature tensor, gives some results on the Pontrjagin classes of a paraholomorphically projectively flat manifold and on the Pontrjagin classes of a para-Kähler space form.

On the other hand, three curvature tensor fields on a para-Kähler manifold are defined in [192, 193]: the *H-conformal curvature tensor*, the *H-concircular curvature tensor* and the *H-conharmonic curvature tensor*. Some relations among them, and some properties of H-concircular flatness are also given.

Definition 6.3. ([192]) Let (M, g, J) be a $2n$-dimensional para-Kähler manifold with curvature R and scalar curvature ρ. The *H-concircular curvature tensor* $\tilde{T}$ is defined by

$$\begin{aligned} \tilde{T}(X,Y,Z) = R(X,Y)Z - \frac{\rho}{4n(n+1)}\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ\}, \quad X,Y,Z \in \mathfrak{X}(M). \end{aligned}$$

The so-called *conformal connection* on a para-Kähler manifold is defined in [190]. The corresponding Bochner tensor is obtained and studied there.

6.2 Geometric vector fields

When one has an almost product structure J on a manifold, the concept of geodesic curve can be generalized to that of paraholomorphically planar curve:

Definition 6.4. ([188]) Let (M, J) be an almost paracomplex manifold endowed with a linear connection ∇. A *paraholomorphically planar curve* $\gamma(t)$ is a curve in M such that

$$\nabla_{\gamma'}\gamma' = f(t)\gamma' + h(t)J\gamma' .$$

where $f(t)$ and $h(t)$ are functions of the parameter t. A *paraholomorphically projective vector field* is a vector field on (M, J) such that its local 1-parameter group of transformations maps each paraholomomorphically planar curve into another.

Paraholomorphically projective vector fields were studied in [78], and from there we recall some of the results obtained:

Theorem 6.5.

(1) *Let (M, J) be a locally product manifold endowed with a torsionless J-connection ∇. A vector field X on (M, J, ∇) is a para-holomorphically projective vector field if and only if:*

(a) $L_X J = 0$;

(b) *There exists an 1-form θ on M such that*

$$(L_X\nabla)(Y,Z) = \theta(Y)Z + \theta(Z)Y + \theta(JY)JZ + \theta(JZ)JY.$$

(2) *Let X be a paraholomorphically projective vector field on a para-Kähler manifold (M^{2n}, g, J). If θ denotes the 1-form associated with X, in the sense of* (b) *in* (1) *above, then* $\theta = (1/(2n+2))d(\operatorname{div} X)$.

(3) *Let X be a paraholomorphically projective vector field on a para-Kähler-Einstein manifold with dimension $2n > 2$ and nonzero scalar curvature ρ, θ the 1-form associated to X, and Δ, $\tilde{\Delta}$, respectively, the flat and the complete Laplacian on M. Then $\tilde{\Delta}\theta = 2\Delta\theta = (\rho/n)\theta$.*

6.3 Other transformations

Dzavadov proves in [68] that the group of *conformal transformations* of the paracomplex space of any dimension, endowed with a canonical metric, is isomorphic to a group of linear fractional transformations.

In [51] one can find a study of the behaviour of many classes of geometric structures with regard to the automorphism on the tensor algebra of the manifold M originated by a nonsingular (1,1) tensor field on M. The author proves that an almost: paracomplex, para-Hermitian or para-Kähler structure is transformed into another structure of the same type, but that if the prefix "almost" is dropped, then if one wants preserve the type, some special conditions must be imposed.

7 Para-Hermitian symmetric spaces and para-Hermitian homogeneous spaces

Para-Hermitian symmetric spaces are a subfamily of pseudo-Riemannian symmetric spaces, having relations with symmetric spaces of Hermitian type – introduced by 'Olaffson and Ørsted [169] and independently by Matsumoto [149], and classified by Doi [61] – and with regular symmetric spaces [168] (see Subsection (7.6)). As a general result, we have that all the para-Hermitian symmetric spaces are diffeomorphic to the cotangent bundle of another Riemannian symmetric space, which is sometimes a Hermitian symmetric space (see Table in 7.2).

7.1 Definitions and first results

Definition 7.1. [127] We say that a connected almost para-Hermitian manifold (M, g, J) is a *para-Hermitian symmetric space*, if for each point $p \in M$ there exists a paraholomorphic isometry $s_p \in \operatorname{Aut}(M, g, J)$, called *the symmetry at* p, such that:

(1) $s_p^2 = id$.

(2) p is an isolated fixed point of s_p.

Proposition 7.2. *Any para-Hermitian symmetric space (M, g, J) is homogeneous para-Kähler, and hence homogeneous symplectic.*

Definition 7.3. Let G be a connected Lie group and H a closed subgroup of G. The coset space G/H is called a *para-Hermitian symmetric coset space* if the following three conditions are satisfied:

(1) There exists an involutive automorphism σ of G such that (G, H, σ) is a symmetric triple; that is, if G_σ denotes the subgroup of all the σ-invariant elements in G, and G_σ^0 the identity component of G_σ, then we have $G_\sigma^0 \subset H \subset G_\sigma$.

(2) There exist a $(1,1)$ tensor field J and a pseudo-Riemannian metric g on M such that (M, g, J) is almost para-Hermitian.

(3) Both g and J are G_σ-invariant.

Proposition 7.4. *Any para-Hermitian symmetric space can be represented as a para-Hermitian symmetric coset space. Conversely, any para-Hermitian symmetric coset space is a para-Hermitian symmetric space.*

Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ and σ an involutive automorphism of $\mathfrak{g}$. If $\mathfrak{h}$ is the set of σ-invariant points in $\mathfrak{g}$, then $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ is called a symmetric triple. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and H a closed subgroup of G with Lie algebra $\mathfrak{h}$. Then we say that the coset space G/H is associated with $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ if σ can be extended to an involutive automorphism – denoted by the same letter σ – of G such that, with the above notations, $G_\sigma^0 \subset H \subset G_\sigma$.

Proposition 7.5. ([127]) *Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be a symmetric triple, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the eigenspace decomposition induced by σ. Suppose that a coset space G/H is associated with $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$. Then G/H is a para-Hermitian symmetric coset space, if and only the following (C_1) is satisfied:*

(C_1) *There exist a linear endomorphism J_0 on $\mathfrak{m}$ and a non-degenerate symmetric bilinear form $\langle\,,\,\rangle$ on $\mathfrak{m}$ such that:*

(1) $J_0^2 = id$.
(2) $[J_0, \mathrm{Ad}_{\mathfrak{m}} H] = 0$.
(3) $\langle J_0 X, Y\rangle + \langle X, J_0 Y\rangle = 0, \; X, Y \in \mathfrak{m}$.
(4) $\langle (\mathrm{Ad}_{\mathfrak{m}} h)X, (\mathrm{Ad}_{\mathfrak{m}} h)Y\rangle = \langle X, Y\rangle, \; X, Y \in \mathfrak{m}, \; h \in H$.

Definition 7.6. ([127]) Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be a symmetric triple and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the eigenspace decomposition induced by σ. Suppose that $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies the following condition (C_2):

(C_2) There exists a linear endomorphism J_0 on $\mathfrak{m}$ and a non-degenerate symmetric bilinear form $\langle\,,\,\rangle$ on $\mathfrak{m}$ such that:

(1) $J_0^2 = id$.
(2) $[J_0, \mathrm{ad}_{\mathfrak{m}} \mathfrak{h}] = 0$.
(3) $\langle J_0 X, Y\rangle + \langle X, J_0 Y\rangle = 0, \; X, Y \in \mathfrak{m}$.
(4) $\langle (\mathrm{ad}\, X)Y_1, Y_2\rangle + \langle Y_1, (\mathrm{ad}\, X)Y_2\rangle = 0, \; X \in \mathfrak{h}, \; Y_1, Y_2 \in \mathfrak{m}$.

Then $\{\mathfrak{g}, \mathfrak{h}, \sigma, J_0, \langle\,,\,\rangle\}$ is called a *para-Hermitian symmetric system*. Moreover, if the pair $\{\mathfrak{g}, \mathfrak{h}\}$ is effective, then it is called an *effective para-Hermitian symmetric system.* We recall that the pair $(\mathfrak{g}, \sigma)$ is called *effective* if the representation $\mathrm{ad}_{\mathfrak{q}} \colon \mathfrak{h} \to \mathrm{End}(\mathfrak{q})$ given by $X \mapsto \mathrm{ad}(X)|_{\mathfrak{q}}$ is faithful.

Proposition 7.7. ([127]) *Let $\{\mathfrak{g}, \mathfrak{h}, \sigma, J_0, \langle\, ,\rangle\}$ be an effective semisimple – taking $\mathfrak{g}$ as being semisimple – para-Hermitian symmetric system. Then there exists a unique element $Z_0 \in \mathfrak{h}$ such that $\mathfrak{h}$ is the centralizer $\mathfrak{c}(Z_0)$ of Z_0 in $\mathfrak{g}$ and $J_0 = ad_{\mathfrak{m}} Z_0$.*

Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple. Consider the following condition (C_3):

(C_3) There exists an element $Z \in \mathfrak{g}$ such that $\mathrm{ad}\, Z$ is a semisimple operator having only real eigenvalues and such that $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(Z)$.

It can be proved that conditions (C_2) and (C_3) are equivalent, and that, with the above notations, the symmetric bilinear form $\langle\, ,\rangle$ coincides with the restriction of the Killing form of $\mathfrak{g}$ to the (-1)-eigenspace $\mathfrak{m}$ of σ in $\mathfrak{g}$.

Kaneyuki and Kozai give in [127] several other important facts concerning para-Hermitian symmetric spaces. We recall some of their results:

Theorem 7.8. *Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple. Then:*

(1) *Let G/H be a coset space associated with the triple. Suppose that G/H is a para-Hermitian symmetric coset space. Then $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies the condition (C_3) and H is an open subgroup of the centralizer $C(Z)$ in G. Conversely:*

(2) *Suppose $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies (C_3). Then there exists a connected Lie group G with Lie algebra $\mathfrak{g}$ such that the coset space $G/C(Z)$ is associated with $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$, where $C(Z)$ is the centralizer of Z in G. Furthermore, for an arbitrary open subgroup H of $C(Z)$, the coset space G/H is a para-Hermitian symmetric space. Moreover, there exists a covering manifold M_0 of a symmetric R-space such that $M = G/H$ is diffeomorphic to the cotangent bundle T^*M_0 of M_0.*

Proposition 7.9. *Let G/H be an affine symmetric coset space. Suppose G is simple. Then there is only one G-invariant paracomplex structure up to sign.*

7.2 Classification and structure of semisimple para-Hermitian symmetric spaces

Every para-Hermitian symmetric space with semisimple group is diffeomorphic to the cotangent bundle of a covering manifold of a Riemannian symmetric space of a particular type, called R-symmetric spaces. So, the para-Hermitian symmetric spaces are candidates to be phase spaces of dynamical systems. Since we have, for instance, the phase space $T^*(SO(3))$ of the rigid solid, we are probably faced with important physical situations.

The infinitesimal classification of para-Hermitian symmetric spaces with semisimple group up to paraholomorphic equivalence is obtained in [127] and [120], and in [127] the following table is given:

$(\mathfrak{g}, \mathfrak{h})$	M_0^*
$(\mathfrak{sl}(m+n,\mathbb{R}), \mathfrak{sl}(m,\mathbb{R}) + \mathfrak{sl}(n,\mathbb{R}) + \mathbb{R})$	$G_{m,n}(\mathbb{R})$
$(\mathfrak{sl}(m+n,\mathbb{C}), \mathfrak{sl}(m,\mathbb{C}) + \mathfrak{sl}(n,\mathbb{C}) + \mathbb{C})$	$G_{m,n}(\mathbb{C})$
$(\mathfrak{su}^*(2m+2n), \mathfrak{su}^*(2m) + \mathfrak{su}^*(2n) + \mathbb{R})$	$G_{m,n}(\mathbb{H})$
$(\mathfrak{su}(n,n), \mathfrak{sl}(n,\mathbb{C}) + \mathbb{R})$	$U(n)$
$(\mathfrak{so}(n,n), \mathfrak{sl}(n,\mathbb{R}) + \mathbb{R})$	$SO(n)$
$(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n) + \mathbb{R})$	$U(2n)/Sp(n)$
$(\mathfrak{so}(2n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C}) + \mathbb{C})$	$SO(2n)/U(n)$
$(\mathfrak{so}(m+1,n+1), \mathfrak{so}(m,n) + \mathbb{R})$	$Q_{m+1,n+1}(\mathbb{R})$
$(\mathfrak{so}(n+2,\mathbb{C}), \mathfrak{so}(n,\mathbb{C}) + \mathbb{C})$.	$Q_n(\mathbb{C})$
$(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}) + \mathbb{R})$	$U(n)/O(n)$
$(\mathfrak{sp}(n,n), \mathfrak{su}^*(2n) + \mathbb{R})$	$Sp(n)$
$(\mathfrak{sp}(n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C}) + \mathbb{C})$	$Sp(n)/U(n)$
$(E_6^1, \mathfrak{so}(5,5) + \mathbb{R})$	$G_{2,2}(\mathbb{H})/\mathbb{Z}_2$
$(E_6^4, \mathfrak{so}(1,9) + \mathbb{R})$	$P_2(\mathbb{O})$
$(E_6^{\mathbb{C}}, \mathfrak{so}(10,\mathbb{C}) + \mathbb{C})$	$E_6/Spin(10) \cdot T^1$
$(E_7^1, E_6^1 + \mathbb{R})$	$SU(8)/Sp(4) \cdot \mathbb{Z}_2$
$(E_7^3, E_6^4 + \mathbb{R})$	$T^1 \cdot E_6/F_4$
$(E_7^{\mathbb{C}}, E_6^{\mathbb{C}} + \mathbb{C})$	$E_7/E_6 \cdot T^1$

In the above list, $G_{m,n}(\mathbb{F})$ denotes the Grassmann manifold of m-planes in $\mathbb{F}^{m+n}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. $Q_{m,n}(\mathbb{R})$ denotes the real quadric in $P_{m+n-1}(\mathbb{R})$ defined by the quadratic form of signature (m,n). $Q_n(\mathbb{C})$ denotes the complex quadric in $P_{n+1}(\mathbb{C})$. $P_2(\mathbb{O})$ denotes the octonion projective plane. The list on the right of the table contains those R-symmetric spaces M_0^* with the property that if $M = G/H$ is a para-Hermitian symmetric space corresponding to the symmetric pair $(\mathfrak{g}, \mathfrak{h})$ associated to the specific M_0^*, then M is diffeomorphic to the cotangent bundle T^*M_0 of a covering manifold M_0 of M_0^*. Note the six Hermitian symmetric spaces, appearing on the right.

Remark 7.10. We recall here the following definitions: Let G (resp. $\tilde{G}$) be a reductive irreducible real (resp. connected complex) algebraic group. The quotient space $M = G/U$ (resp. $\tilde{M} = \tilde{G}/\tilde{U}$) by a parabolic subgroup U (resp. $\tilde{U}$) of G (resp. $\tilde{G}$) is called a R-space (resp. complex R-space). For the definition of a symmetric R-space, which involves Dynkin diagrams, see, for instance, [242, p. 82].

The structure of the (simple) group G of a para-Hermitian symmetric space $M = G/H$ is studied in [122], where Kaneyuki obtains a decomposition of G for the case in which the Weyl group $W(M)$ coincides with the Weyl group $W(M^*)$ of the fiber M^* of the Berger fibration of M (see [83]). That condition is not too restrictive. This decomposition – with intersection – is $G = KCH_l$ $(0 \le l \le r = \dim C)$, where K is a σ-stable maximal compact subgroup of G– and σ denotes the involutive automorphism associated to G –, C a split Cartan subgroup of G, H_0 the isotropy group of G at a point in M, and H_l $(1 \le l \le r)$ the isotropy group of G on a point of the boundary of M in Kaneyuki's compactification $\tilde{M}$ of M. This is a partial generalization of a result given in [83] and [219] on the decomposition of the group G of a semisimple affine symmetric space $M = G/H$.

The isotropy group of a para-Hermitian symmetric space was studied in [128]. The number of connected components and the structure of the identity component is obtained:

Theorem 7.11. *Suppose $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ is an effective simple – that is, $\mathfrak{g}$ is simple – symmetric triple which satisfies condition (C_3) in Section 7.1 for some $Z \in \mathfrak{g}$. Let $G/C(Z)$ be the coset space associated with $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ in Theorem 7.8. Then:*

(1) *If $\mathfrak{g}$ does not admit a complex Lie algebra structure, then $C(Z)$ has at most 2 connected components, and the identity component $C^0(Z)$ has a smooth direct product decomposition $C^0(Z) = S(Z) \cdot \mathbb{R}^+(Z)$, where $S(Z)$ denotes the (semisimple) commutator subgroup $[C^0(Z), C^0(Z)]$ and $\mathbb{R}^+(Z) = \exp \mathbb{R}Z$ is the center of $C^0(Z)$.*

(2) *If $\mathfrak{g}$ admits a complex Lie algebra structure which commutes with σ, then $C(Z)$ is connected and has a smooth direct product decomposition given by $C(Z) = S'(Z) \cdot \mathbb{R}^+(Z)$, where $S'(Z) = [C(Z), C(Z)] \exp \mathbb{R}\, iZ$ and $\mathbb{R}^+(Z) = \exp \mathbb{R}Z$.*

As is well-known, M. Flensted-Jensen's knowledge of both the theories of group representations and semisimple symmetric spaces, has permitted him to emphasize the important role of the affine symmetric spaces in the theory of group representations (see [31, 84, 177]). In this spirit, Kaneyuki studies in [121] the orbit structure of compactifications of para-Hermitian symmetric spaces. We recall here the example given by him, which gives an idea of the general situation: Let $\mathcal{H}$ be the hyperboloid of revolution in $\mathbb{R}^3$ given by the equation $x^2+y^2-z^2 = 1$. $\mathcal{H}$ is viewed as the cotangent bundle of the real projective space $P_1(\mathbb{R})$. $\mathcal{H}$ is written as the affine symmetric space $SL(2,\mathbb{R})/\mathbb{R}^*$, where $\mathbb{R}^*$ is identified with the subgroup of diagonal matrices in $SL(2,\mathbb{R})$. The $SL(2,\mathbb{R})$-action on $\mathcal{H}$ leaves invariant the two families L_1 and L_2 of generatrices of $\mathcal{H}$, respectively. Through an arbitrary point $p \in \mathcal{H}$ there pass two generating lines $l_i \in L_i$ $(i = 1, 2)$, which meet the line of stricture of $\mathcal{H}$, viewed as $P_1(\mathbb{R})$, in two points q_i. By assigning the pair (q_1, q_2) to the point $p \in \mathcal{H}$ we have an embedding of $\mathcal{H}$ into the 2-torus $T^2 = P_1(\mathbb{R}) \times P_1(\mathbb{R})$. The embedding maps L_1 or L_2 into the meridians or the parallels on T^2, respectively. The $SL(2,\mathbb{R})$-action on $\mathcal{H}$ is hence transferred to the action on T^2 of the diagonal subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. An elementary argument shows that the torus T^2 is decomposed in two $SL(2,\mathbb{R})$-orbits: one is $\mathcal{H}$ and the other is a 1-dimensional orbit diffeomorphic to $P_1(\mathbb{R})$. In [122], this phenomenon is generalized to higher dimensions: *Let $M = G/H$ be a para-Hermitian symmetric space, which is diffeomorphic to the cotangent bundle of a covering manifold of a symmetric R-space M_0^*. Let $\tilde{M} = M_0^* \times M_0^*$. Then, M is imbedded in $\tilde{M}$ as a single orbit through the origin of $\tilde{M}$ under the action of the diagonal subgroup of $G \times G$. Thus $\tilde{M}$ can be viewed as a compactification of M.* In [122] the orbit structure of $\tilde{M}$ is studied. It turns out that it is somewhat similar to the structure of the closure of an irreducible bounded symmetric domain under its holomorphic automorphism group [135]. This situation extends to the case of para-Hermitian homogeneous spaces (see Subsection 7.7).

7.3 Para-Grassmannian manifolds

These are the para-Hermitian symmetric spaces corresponding to the symmetric pairs $(\mathfrak{sl}(m+n, \mathbb{F}), \mathfrak{sl}(m, \mathbb{F}) + \mathfrak{sl}(n, \mathbb{F}) + \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, in Kaneyuki-Kozai's classification. These manifolds were studied by Kaneyuki, Kozai and Williams, and also – in the real case – in [89], where they are named para-Grassmannian manifolds because they are spaces whose points are non-degenerate subspaces. Their para-Kähler structure in the real and complex cases is explicitly given in [97].

There the authors obtain a pseudo-Riemannian metric of Wong type [256], which is more general than a Fubini-Study metric. Specifically, they state the following result (for more details see the quoted paper):

Theorem 7.12. *Let E be an $(m+n)$-dimensional $\mathbb{F}$-vector space, where $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, and E^* its dual space. Then the C^ω manifolds underlying the homogeneous spaces*

$$G_{m,n}(E\oplus E^*)_{\mathbb{F}} \approx SL(m+n,\mathbb{F})/S(GL_0(m,\mathbb{F})\times GL_0(n,\mathbb{F})) \qquad \text{and}$$
$$G_{m,n}(E\oplus E^*)_{\mathbb{R}}/\mathbb{Z}_2 \approx SL(m+n,\mathbb{R})/S(GL(m,\mathbb{R})\times GL(n,\mathbb{R})),$$

endowed with the para-Kähler structure given by the pseudo-Riemannian metric $g_{\mathbb{R}}$ and the almost product structure $J_{\mathbb{R}}$, given in the charts (x_a^u, y_a^u) ([95], formula (2)), *respectively by*

$$g_{\mathbb{R}} = \operatorname{Re}\operatorname{Tr}\left[(I+Y^t\cdot X)^{-1}\{dY^t\cdot dX - dY^t\cdot X\cdot(I+Y^t\cdot X)^{-1}\cdot Y^t\cdot dX\}\right],$$

where $X=(x_a^u)$, $Y=(y_a^u)$, and

$$J_{\mathbb{R}}(X) = \operatorname{Re}\left(\frac{\partial}{\partial x_a^u}\otimes dx_a^u - \frac{\partial}{\partial y_a^u}\otimes dy_a^u\right)(X - iI(X)),$$

where I denotes the almost complex structure associated with the given complex manifold, are the para-Hermitian symmetric spaces corresponding to the two families $(\mathfrak{sl}(m+n,\mathbb{F}), \mathfrak{sl}(m,\mathbb{F})+\mathfrak{sl}(n,\mathbb{F})+\mathbb{F})$, where $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, in Kaneyuki-Kozai's infinitesimal classification of para-Hermitian symmetric spaces with semisimple group.

Probably the main interest of the above result is that the knowledge of the para-Kähler structure in those cases can be useful for knowing the para-Kähler structure in the other 16 cases in Kaneyuki-Kozai's classification.

A particular case of para-Grassmannians are the so-called *paracomplex models* $P_n(\mathbb{B})$ (see Section 8.2), which are models of spaces of non vanishing constant paraholomorphic sectional curvature, similar in some aspects to the complex projective spaces $P_n(\mathbb{C})$. For instance, one has the diffeomorphism $P_n(\mathbb{B}) \approx SL(n+1,\mathbb{R})/S(GL_0(n,\mathbb{R})\times GL_0(\mathbb{R}))$, while we have $P_n(\mathbb{C}) \approx SU(n+1)/S(U(n)\times U(1))$, and, as is well known, both $SL(n,\mathbb{R})$ and $SU(n)$ are real forms of the complex group $SL(n,\mathbb{C})$. $P_n(\mathbb{B})$ is the "real equivalent" of $P_n(\mathbb{C})$, in Berger's sense [30].

Nevertheless, both spaces are quite different in other aspects. For example, we have a diffeomorphism $P_n(\mathbb{B}) \approx T(S^n)$, and moreover, although $P_n(\mathbb{C})$ has no Hermitian space forms [254], $P_n(\mathbb{B})$ has a rich family of para-Kähler space forms (see [94, 95] and Subsection 8.3).

7.4 Para-Hodge manifolds

Definition 7.13. ([130]) A para-Kähler manifold (M,g,J) is called a *para-Hodge manifold* if the cohomology class $[F]$ of its fundamental 2-form F is an integral class in $H^2(M,\mathbb{R})$.

Example 7.14. ([130]) A para-Hermitian symmetric space with second Betti number $b_2=0$ is always para-Hodge. Let M be the cotangent bundle T^*M_0 over a symmetric R-space M_0. Then M is a para-Hermitian symmetric coset space of a semisimple Lie group. The para-Kähler metric g of M is then induced by the Killing form of the Lie algebra of G. If M_0 is one of the group manifolds $SO(n)$, $U(n)$, $Sp(n)$, $U(2n)/Sp(n)$, or the sphere S^n $(n\geq 2)$, then the second Betti number of M vanishes, and so g is para-Hodge.

Example 7.15. ([130]) Let (M, g, J) be a para-Hermitian symmetric space with simple automorphism group $\mathrm{Aut}(M, g, J)$, where g is induced from the Killing form of G. Then (M, g, J) is para-Hodge with $b_1(M) = 0$ if and only if M is the cotangent bundle of a covering manifold of a symmetric R-space M_0^*, which is not the Silov boundary of an irreducible symmetric bounded domain.

For an effective semisimple triple $\{(\mathfrak{g}, \mathfrak{h}), \sigma\}$ let us consider the following:

Definition 7.16. (Condition (C)) There exists an element $Z \in \mathfrak{g}$ satisfying:

(1) $\mathrm{ad}\, Z$ is a semisimple operator with eigenvalues $0, \pm 1$ only.

(2) $\mathfrak{h}$ coincides with the centralizer of Z in $\mathfrak{g}$.

Theorem 7.17. ([130]) *Let $\{(\mathfrak{g}, \mathfrak{h}), \sigma\}$ be a simple symmetric triple satisfying the condition* (C) *in* 7.16. *Let $\tilde{G}$ be the simply connected Lie group corresponding to $\mathfrak{g}$. Suppose that the centralizer $\tilde{C}(Z)$ in $\tilde{G}$ is connected, or equivalently, that $\tilde{G}/\tilde{C}(Z)$ is simply connected. Then the para-Kähler metric of the para-Hermitian symmetric coset space $\tilde{G}/\tilde{C}(Z)$ induced by the Killing form of $\mathfrak{g}$ is para-Hodge.*

Corollary 7.18. ([130]) *Suppose G is simple. Then the simply connected coset spaces $G/C(Z)$ with a para-Hodge structure induced by the Killing form of $\mathfrak{g}$ are given up to infinitesimal equivalence by the pairs $\{\mathfrak{g}, \mathfrak{h}\}$ – in the symmetric triple $\{(\mathfrak{g}, \mathfrak{h}), \sigma\}$ – as follows:*

(1) *With* $b_2 = 0$	(2) *With* $b_2 \neq 0$
$(\mathfrak{su}^*(2m+2n), \mathfrak{su}^*(2m) + \mathfrak{su}^*(2n) + \mathbb{R})$	$(\mathfrak{sl}(m+n, \mathbb{C}), \mathfrak{sl}(m, \mathbb{C}) + \mathfrak{sl}(n, \mathbb{C}) + \mathbb{C})$
$(\mathfrak{so}(n+1, 1), \mathfrak{so}(n) + \mathbb{R})$ *for* $n \geq 2$	$(\mathfrak{so}(2n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}) + \mathbb{C})$
$(\mathfrak{sp}(n, n), \mathfrak{su}^*(2n) + \mathbb{R})$	$(\mathfrak{so}(n+2, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}) + \mathbb{C})$
$(E_6^4, \mathfrak{so}(1, 9) + \mathbb{R})$	$(\mathfrak{sp}(n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}) + \mathbb{C})$
	$(E_6^{\mathbb{C}}, \mathfrak{so}(10, \mathbb{C}) + \mathbb{C})$
	$(E_7^{\mathbb{C}}, E_6^{\mathbb{C}} + \mathbb{C})$

In [130] more examples of para-Hodge manifolds are given, including for instance the real para-Grassmannians.

7.5 Semisimple para-Hermitian symmetric spaces as quantizable coadjoint orbits

Definition 7.19. A symplectic manifold (M, ω) is said to be *quantizable* if the cohomology class $[\omega]$ in $H^2(M, \mathbb{R})$ of the non-degenerate closed 2-form ω lies in the image $i^* H^2(M, \mathbb{Z})$ of the homomorphism $i^* \colon H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$ induced by the inclusion map $i \colon \mathbb{Z} \hookrightarrow \mathbb{R}$.

The following theorem gives a useful criterion in order to know whether a symplectic manifold is quantizable:

Theorem 7.20. ([140, 252]) *A symplectic manifold (M, ω) is quantizable if and only there is a Hermitian line bundle $L \longrightarrow M$ over M with an invariant connection ∇ such that its curvature form R_∇ satisfies $(1/2\pi i) R_\nabla = \omega$.*

Definition 7.21. Let G be any Lie group. Then G acts on the (real) dual space $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$ via the contragredient of the adjoint representation by

$$(s \circ f)(X) = f(\mathrm{Ad}(s^{-1})X), \qquad s \in G, \quad X \in \mathfrak{g}, \quad f \in \mathfrak{g}^* . \tag{7.1}$$

Each orbit in $\mathfrak{g}^*$ induced by this action is called a *coadjoint orbit.*

Proposition 7.22. ([134]) *Each coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$ of the action of the Lie group G on $\mathfrak{g}^*$ is endowed with a natural closed 2-form ω_0 such that $(\mathcal{O}, \omega_0)$ is a symplectic manifold.*

Let $\{(\mathfrak{g}, \mathfrak{h}), \sigma\}$ be an effective semisimple symmetric triple and $Z \in \mathfrak{g}$ such that:

$$(1) \quad \mathfrak{h} = \{X \in \mathfrak{g} : [X, Z] = 0\} \tag{7.2}$$

and

$$(2) \quad \mathrm{ad}\, Z \colon \mathfrak{g} \longrightarrow \mathfrak{g} \quad \text{has only real eigenvalues.} \tag{7.3}$$

By Theorem 7.8, $G/C(Z)$ is a para-Hermitian symmetric coset space. If B denotes the Killing form of $\mathfrak{g}$, the associated natural isomorphism $\flat \colon \mathfrak{g} \to \mathfrak{g}^*$ is given by $X \mapsto f_X$, where $f_X(Y) = B(Y, X)$ for $Y \in \mathfrak{g}$. This isomorphism intertwines the adjoint action of G on G and the coadjoint action of G on $\mathfrak{g}^*$ given in (7.1). Thus $\mathcal{O}_Z = G/C(Z)$ is a coadjoint orbit. Then we have:

Theorem 7.23. ([129]) *Let $\mathcal{O}_Z = G/C(Z)$ be the para-Hermitian symmetric coadjoint orbit associated with the effective semisimple triple $\{(\mathfrak{g}, \mathfrak{h}), \sigma\}$ which satisfies* (7.2) *and* (7.3), *and ω_Z the Kirillov structure on $\mathcal{O}_Z$ (Proposition* 7.22*). Suppose G is simple and does not admit a complex Lie algebra structure, and suppose $C(Z)$ is connected. Then $(\mathcal{O}_Z = G/C(Z), \omega_Z)$ is quantizable. Moreover, $G/C(Z)$ is a para-Hodge manifold.*

Theorem 7.24. ([129]) *Let $(\mathfrak{g}, \mathfrak{h})$ denote one of the following symmetric pairs: $(\mathfrak{su}^*(2m+2n), \mathfrak{su}^*(2m) + \mathfrak{su}^*(2n) + \mathbb{R})$,(the quaternionic case); $(\mathfrak{so}(n+1, 1), \mathfrak{so}(n) + \mathbb{R})$ for $n \geq 2$, (the n-sphere case); $(\mathfrak{sp}(n, n), \mathfrak{su}^*(2n) + \mathbb{R})$, (the Sp(n)-case); and $(E_6^4, \mathfrak{so}(1, 9) + \mathbb{R})$, (the octonion case). Then $(\mathfrak{g}, \mathfrak{h})$ is part of an effective simple triple (i.e., $\mathfrak{g}$ is simple), which satisfies* (7.2) *and* (7.3) *for a suitable $Z \in \mathfrak{g}$ such that the orbit $\mathcal{O}_Z$ in Theorem* 7.23 *is simply connected. In particular $C(Z)$ is connected and all of the conclusions of Theorem* 7.23 *apply to $(\mathcal{O}_Z, \omega_Z)$.*

The condition "$C(Z)$ connected" in the previous theorem can be weakened under certain conditions and we can conclude [130] that, for instance, the real para-Grassmannians associated to $(\mathfrak{sl}(m+n, \mathbb{R}), \mathfrak{sl}(m, \mathbb{R}) + \mathfrak{sl}(n, \mathbb{R}) + \mathbb{R})$ are quantizable coadjoint orbits.

The following result gives us more examples related to compact irreducible Hermitian symmetric spaces:

Theorem 7.25. ([129]) *Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective simple triple which satisfies* (7.2) *and* (7.3) *and such that $\mathfrak{g}$ admits a complex Lie algebra structure which commutes with σ. Let $\mathcal{O}_Z = G/C(Z)$ be the para-Hermitian symmetric coadjoint orbit associated with $\{(\mathfrak{g}, \mathfrak{h}), \sigma\}$ and ω_Z the associated Kirillov symplectic 2-form. Then $(\mathcal{O}_Z = G/C(Z), \omega_Z)$ is quantizable. Moreover, for each of the following pairs $\{\mathfrak{g}, \mathfrak{h}\}$ there is a σ such that $\{(\mathfrak{g}, \mathfrak{h}), \sigma\}$ satisfies the preceding hypotheses: $(\mathfrak{sl}(m+n, \mathbb{C}), \mathfrak{sl}(m, \mathbb{C}) + \mathfrak{sl}(n, \mathbb{C}) + \mathbb{C})$, $(\mathfrak{so}(2n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}) + \mathbb{C})$, $(\mathfrak{so}(n+2, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}) + \mathbb{C})$, $(\mathfrak{sp}(n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}) + \mathbb{C})$, $(E_6^{\mathbb{C}}, \mathfrak{so}(10, \mathbb{C}) + \mathbb{C})$ or $(E_7^{\mathbb{C}}, E_6^{\mathbb{C}} + \mathbb{C})$.*

The corresponding quantizable orbit $\mathcal{O}_Z = G/C(Z)$ is simply connected and is diffeomorphic to the cotangent bundle of a compact irreducible Hermitian symmetric space.

7.6 Symmetric spaces of Hermitian type

As Hilgert, 'Olafsson and Ørsted have proved, there is a narrow link between symmetric spaces of Hermitian type, Kaneyuki-Kozai's para-Hermitian symmetric spaces and Ol'shankii's regular symmetric spaces. Concerning this, we now recall some definitions and results.

Let (G, H, σ) be a semisimple symmetric pair, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$. Suppose for the sake of simplicity that G is contained in the simply connected group $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. There always exists a Cartan involution τ commuting with σ. H is an open subgroup of the fixpoint subgroup G_σ. Let $K = G_\tau$ be the fixpoint group of τ in G. Then we have an orthogonal – with respect to the inner product $(X,Y)_\tau := -\operatorname{Tr}(\operatorname{ad}(X)\operatorname{ad}(\tau Y))$ – direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h}_k \oplus \mathfrak{h}_p \oplus \mathfrak{m}_k \oplus \mathfrak{m}_p\,,$$

where $\mathfrak{h}{=}\mathfrak{g}_\sigma$ is the Lie algebra of H, $\mathfrak{k}{=}\mathfrak{g}_\tau$ is the Lie algebra of K, $\mathfrak{m}{:=}\mathfrak{h}^\perp{=}\{X \in \mathfrak{g} : \sigma(X){=}-X\}$, $\mathfrak{p} := \mathfrak{k}^\perp = \{X \in \mathfrak{g} : \tau(X) = -X\}$ and the subscript k (resp. p) denotes the intersection with $\mathfrak{k}^{\mathbb{C}}$ (resp. $\mathfrak{p}^{\mathbb{C}}$), where the superscript $\mathbb{C}$ denotes the complexification. Let $D := G/K$ and $M := G/H$. Then D is a Riemannian symmetric space and M is a pseudo-Riemannian symmetric space. Let $\mathfrak{c}(\mathfrak{m}_k)$ be the *center* of $\mathfrak{m}_k$, i.e., $\mathfrak{c}(\mathfrak{m}_k) = \{X \in \mathfrak{m}_k : [X,Y] = 0,\ \forall Y \in \mathfrak{m}_k\}$. The pair $(\mathfrak{g}, \sigma)$ is said to be of *Hermitian type* if $\mathfrak{z}_{\mathfrak{m}}(\mathfrak{c}(\mathfrak{m}_k)) = \mathfrak{m}_k$, and there is no non-trivial, non-compact ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. We say that M and σ are of Hermitian type if $(\mathfrak{g}, \mathfrak{h})$ is of Hermitian type.

The definition of a *para-Hermitian symmetric pair* is Definition 7.6.

The regular spaces are defined by interchanging the rôle of the compact and non-compact part of $\mathfrak{m}$. For $\mathfrak{g}$ simple those spaces were first introduced by Ol'shanskii [170, 171].

The semisimple symmetric pair $(\mathfrak{g}, \sigma)$ is called *regular* if $\mathfrak{z}_{\mathfrak{m}}(\mathfrak{c}(\mathfrak{m}_p)) = \mathfrak{m}_p$ where $\mathfrak{c}(\mathfrak{m}_p)$ is the center of $\mathfrak{m}_p$.

View $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ and let κ be the conjugation of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{g}$. Define

$$\mathfrak{g}^c := \mathfrak{g}^{\mathbb{C}}_{\sigma^c} = \mathfrak{h} \oplus i\mathfrak{m} \qquad \text{and} \qquad \mathfrak{g}^r := \mathfrak{g}^{\mathbb{C}}_{\tau^c\sigma^c} = \mathfrak{h}_k \oplus i\mathfrak{h}_p \oplus \mathfrak{m}_p \oplus i\mathfrak{m}_k\,.$$

We write $(\mathfrak{g},\sigma)^c := \mathfrak{g}^c$ and $(\mathfrak{g},\sigma)^r := \mathfrak{g}^r$. By holomorphic extension and restriction, σ and τ define involutions on $\mathfrak{g}^c$ and $\mathfrak{g}^r$. Those involutions are denoted by the same letters or with the superscript ^{c} (respectively ^{r}). Then $\sigma^c = \kappa|^c_{\mathfrak{g}}$, $\sigma^c\tau^c =: \tau_0$ is a Cartan involution of $\mathfrak{g}^c$, σ^r is a Cartan involution of $\mathfrak{g}^r$ and $\tau\sigma|^r_{\mathfrak{g}} = \kappa|_{\mathfrak{g}^r}$. The *associated pair* is defined by $(\mathfrak{g},\sigma)^a := (\mathfrak{g}, \sigma\tau)$, and we let $\sigma^a = \sigma\tau$. Notice that ^{r} and ^{c} are related by $(\mathfrak{g},\sigma,\tau)^r = (\mathfrak{g},\sigma^a,\tau)^c$, with the obvious notation. The pair $(\mathfrak{g},\sigma)^c$ is called the *c-dual* of $(\mathfrak{g},\sigma)$ and $(\mathfrak{g},\sigma)^r$ is the *dual* or *Riemannian dual* of $(\mathfrak{g},\sigma)$.

Theorem 7.26. ([167]) *Let $(\mathfrak{g},\sigma)$ be an effective symmetric pair such that $\mathfrak{g}$ has no compact ideals. Then the following assertions are equivalent:*

(1) $(\mathfrak{g},\sigma)$ *is of Hermitian type.*

(2) $(\mathfrak{g}^c,\sigma)$ *is regular.*

(3) $(\mathfrak{g}^c,\tau)$ *is effective and para-Hermitian.*

(4) $(\mathfrak{g},\tau\sigma)$ *is of Hermitian type.*

(5) $(\mathfrak{g}^r,\tau\sigma)$ *is regular.*

(6) $(\mathfrak{g}^r,\tau)$ *is effective and para-Hermitian.*

Now, let L be a Lie group and $(V, \langle\,,\,\rangle)$ a finite dimensional real Euclidean vector space and an L-module. A subset $C \subset V$ is an *open (convex) cone* if C is open, (convex) and $(\mathbb{R}^+ - \{0\})C \subset C$. If C is an open cone we define the *dual cone* C^* by $C^* := \{u \in V : \langle u, v\rangle > 0, \forall v \in \bar{C} - \{0\}\}$. C is *proper* if both C and C^* are non-zero.

Let C be an open and proper convex cone in a real vector space V. Then

$$D(C) := V + iC \subset V^{\mathbb{C}}$$

is called a *tube domain over* C and also a *Siegel domain of type* I.

If G/K is of tube type, then a *Cayley transform* of G/K is a map $\mathcal{C}$ of G/K to a tube domain $D(C)$, defined by $\mathcal{C} = \mathrm{Ad}\left(\exp \frac{1}{2}\pi i X\right)$, where $\mathrm{ad}(X)$ is supposed to have eigenvalues $0, 1, -1$.

Theorem 7.27. ([167]) *Let* $\mathfrak{g}$ *be simple and* σ *of Hermitian type. Then the following assertions are equivalent:*

(1) G/K *is a tube domain and there exists a Cayley transform* $\mathcal{C}$ *such that* σ *is conjugate to* $\mathcal{C}^2$.

(2) $\mathfrak{m}$ *is reducible as* $\mathfrak{h}$*-module.*

(3) *If* $\mathfrak{c}(\mathfrak{h})$ *is the center of* $\mathfrak{h}$*, then* $\dim \mathfrak{c}(\mathfrak{h}) = 1$ *and* $\mathfrak{c}(\mathfrak{h}) \subset \mathfrak{p}$.

(4) σ *is inner and* $\sigma = \sigma_{\mathfrak{p}}$.

(5) $\sigma = \mathrm{Ad}(\exp X)$ *is inner and* $\mathfrak{h} \subset \mathfrak{z}_{\mathfrak{g}}(X)$.

(6) *All the spaces in Theorem 7.26 are isomorphic.*

(7) $(\mathfrak{g}, \sigma)$ *is isomorphic to one of the pairs* $(\mathfrak{g}^c, \sigma)$, $(\mathfrak{g}^c, \tau)$, $(\mathfrak{g}^r, \tau)$, *or* $(\mathfrak{g}^r, \sigma^a)$.

(8) $(\mathfrak{g}, \sigma)$ *is regular.*

(9) $(\mathfrak{g}, \sigma)$ *is para-Hermitian.*

(10) $(\mathfrak{g}^r, \tau)$ *is of Hermitian type.*

The symmetric spaces which satisfy these 10 equivalent conditions are called by 'Olafsson *symmetric spaces of Cayley type*, because of the intervention of the Cayley transform.

7.7 Para-Hermitian homogeneous spaces

These spaces have been introduced and studied in [123] (see also [124, 126]). Kaneyuki obtains a classification in terms of a natural number ν in such a way that para-Hermitian symmetric spaces correspond to the case $\nu = 1$. This number is associated with the kind of GLA (graded Lie algebra) involved. We now give some specific definitions and results on para-Hermitian homogeneous spaces, including their relation with three Lie algebra objects: para-Kähler algebras, dipolarizations and graded Lie algebras.

Definition 7.28. Let G be a connected Lie group and H a closed subgroup of G. Suppose that the coset space G/H has a para-Kähler structure (g, J). If G leaves both g and J invariant, then G/H is called a *para-Kähler coset space.*

Definition 7.29. Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$, J a linear endomorphism of $\mathfrak{g}$ and ρ an alternating 2-form on $\mathfrak{g}$. Then the cuadruple $\{\mathfrak{g}, \mathfrak{h}, J, \rho\}$ is called a *para-Kähler algebra,* if the following conditions are satisfied:

(1) $J(\mathfrak{h}) \subset \mathfrak{h}$ and $J^2 \equiv 1 \pmod{\mathfrak{h}}$. The $\pm$-eigenspaces of the operator on the space $\mathfrak{g}/\mathfrak{h}$ induced by J are equi-dimensional.

(2) $[X, JY] \equiv J[X, Y] \pmod{\mathfrak{h}}, \quad X \in \mathfrak{h}, \quad Y \in \mathfrak{g}$.

(3) $[JX, JY] \equiv J[JX, Y] + J[X, JY] - [X, Y] \pmod{\mathfrak{h}}, \quad X, Y \in \mathfrak{g}$.

(4) $\rho(X, \mathfrak{g}) = 0 \Leftrightarrow X \in \mathfrak{h}$.

(5) $\rho(JX, JY) = -\rho(X, Y), \quad X, Y \in \mathfrak{g}$.

(6) $\rho([X, Y], Z) + \rho([Y, Z], X) + \rho([Z, X], Y) = 0, \quad X, Y, Z \in \mathfrak{g}$.

If the 2-form ρ is a coboundary df of a linear form in the sense of the Lie algebra cohomology, then the para-Kähler algebra $\{\mathfrak{g}, \mathfrak{h}, J, \rho\}$ is called a *nondegenerate para-Kähler algebra.* In this case (4)-(6) above can be replaced by:

(7) $f([X, \mathfrak{g}]) = 0 \Leftrightarrow X \in \mathfrak{h}$.

(8) $f([JX, JY]) = -f([X, Y]), \quad X, Y \in \mathfrak{g}$.

Theorem 7.30. *Let G be a connected Lie group and H a closed subgroup of G, whose Lie algebras are $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Under these hypotheses:*

(1) *Suppose that G/H is a para-Kähler coset space. Then there exist a linear endomorphism J of $\mathfrak{g}$ and an alternating 2-form ρ on $\mathfrak{g}$ such that $\{\mathfrak{g}, \mathfrak{h}, J, \rho\}$ is a para-Kähler algebra.*

(2) *Suppose that the pair $\{\mathfrak{g}, \mathfrak{h}\}$ has the structure of a para-Kähler algebra given by $\{\mathfrak{g}, \mathfrak{h}, J, \rho\}$. Suppose furthermore that:*

   (a) $[\mathrm{Ad}\, a, J] \equiv 1 \pmod{\mathfrak{h}}, \ a \in H$.

   (b) $\rho((\mathrm{Ad}\, a)X, (\mathrm{Ad}\, a)Y) = \rho(X, Y), \ a \in H, \ X, Y \in \mathfrak{g}$.

   Then G/H is a para-Kähler coset space. This assertion holds if H is connected, without assumming (a) *and* (b).

Definition 7.31. Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{g}^{\pm}$ two subalgebras of $\mathfrak{g}$ and ρ an alternating 2-form on $\mathfrak{g}$. The triple $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ is called a *weak dipolarization in* $\mathfrak{g}$, if the following conditions are satisfied:

(1) $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$.

(2) Put $\mathfrak{h} := \mathfrak{g}^+ \cap \mathfrak{g}^-$. Then $\rho(X, \mathfrak{g}) = 0 \Leftrightarrow X \in \mathfrak{h}$.

(3) $\rho(\mathfrak{g}^+, \mathfrak{g}^+) = \rho(\mathfrak{g}^-, \mathfrak{g}^-) = 0$.

(4) $\rho([X, Y], Z) + \rho([Y, Z], X) + \rho([Z, X], Y) = 0, \ X, Y, Z \in \mathfrak{g}$.

It follows from (1)-(3) that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are equi-dimensional.

Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{g}^{\pm}$ two subalgebras of $\mathfrak{g}$ and f a linear form on $\mathfrak{g}$. The triple $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is called a *dipolarization in* $\mathfrak{g}$, if the following conditions are satisfied:

(1) $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$.

(2) Put $\mathfrak{h} := \mathfrak{g}^+ \cap \mathfrak{g}^-$. Then $f([X, \mathfrak{g}]) = 0 \Leftrightarrow X \in \mathfrak{h}$.

(3) $f([\mathfrak{g}^+, \mathfrak{g}^+]) = f([\mathfrak{g}^-, \mathfrak{g}^-]) = 0$.

A weak dipolarization can be obtained from a dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ just by taking df as ρ.

Theorem 7.32. ([124]) *Let $\mathfrak{g}$ be a real Lie algebra. Then there exists a bijection between the set of isomorphism classes of para-Kähler algebra structures on $\mathfrak{g}$ and the set of isomorphism classes of weak dipolarizations in $\mathfrak{g}$.*

Theorem 7.33. ([124]) *Let G be a connected Lie group and H a closed subgroup of G, with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Suppose that G/H is a para-Kähler coset space. Then $\mathfrak{g}$ admits a dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ such that*

(1) $\mathfrak{h} = \mathfrak{g}^+ \cap \mathfrak{g}^-$.

Conversely, suppose that there exists a weak dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ in $\mathfrak{g}$ satisfying the conditions (1) *and*

(2) $(\mathrm{Ad}_{\mathfrak{g}} H)\mathfrak{g}^\pm \subset \mathfrak{g}^\pm$;

(3) ρ *is* $(\mathrm{Ad}_{\mathfrak{g}} H)-$*invariant.*

Then G/H is a para-Kähler coset space.

The above manifold G/H is called the *para-Kähler coset space corresponding to the weak dipolarization* $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$.

Definition 7.34. Let $\mathfrak{g}$ be a Lie algebra and $Z^0 \in \mathfrak{g}$. Then $(\mathfrak{g}, Z^0)$ is called a *graded Lie algebra of the ν-th kind* if $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$, where $\mathfrak{g}_k = \{X \in \mathfrak{g} : \mathrm{ad}(Z^0)X = kX\}$. Z^0 is called the *characteristic element* of the GLA $\mathfrak{g}$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and B the Killing form of $\mathfrak{g}$. Note that in this case a weak dipolarization in $\mathfrak{g}$ is always a polarization, since the second cohomology group of $\mathfrak{g}$ vanishes.

Proposition 7.35. ([124]) *Let $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ be a dipolarization in $\mathfrak{g}$. Then $\mathfrak{h} := \mathfrak{g}^+ \cap \mathfrak{g}^-$ coincides with the centralizer $\mathfrak{c}(Z)$ in $\mathfrak{g}$ of an element $Z \in \mathfrak{g}$.*

Theorem 7.36. *Let $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ be a semisimple graded Lie algebra of the ν-th kind, and $Z \in \mathfrak{g}$ its characteristic element. Let $\mathfrak{g}^\pm = \sum_{k=0}^{\nu} \mathfrak{g}_{\pm k}$. Define a linear form f on $\mathfrak{g}$ by $f(X) = B(Z, X)$, $X \in \mathfrak{g}$. Then $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is a dipolarization in $\mathfrak{g}$.*

This dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is called the *canonical dipolarization* in the GLA $\mathfrak{g}$.

Theorem 7.37. *Let $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ be a semisimple graded Lie algebra of the ν-th kind, and $Z \in \mathfrak{g}$ its characteristic element. Let G be a connected Lie group generated by $\mathfrak{g}$ and $C(Z)$ the centralizer of Z in G. Then $M := G/C(Z)$ is a para-Kähler coset space.*

This para-Kähler coset space $G/C(Z)$ is called a *semisimple para-Kähler coset space of the ν-th kind.* If G is simple, then it is called a *simple para-Kähler coset space.*

Remark 7.38. Deng and Kaneyuki give in [58] an example of nonsymmetric dipolarizations on the Lie algebra of upper triangular matrices, and pose the following problem: Are there nonsymmetric dipolarizations on semisimple Lie algebras? If so, classify them.

Theorem 7.39. [112] *A homogeneous space G/H, where G is a connected semisimple Lie group and H is a closed subgroup of G, admits a G-invariant para-Kähler structure if and only if H is the identity component of a centralizer of a noncompact abelian subgroup of G.*

A characterization of these invariant structures in terms of root systems related to G and H is given in [112]. Somewhat specifically, $\Delta(G)$ and $\Delta(H)$ being the roots systems of G and H, the author characterizes those structures by partitions of the complementary roots $\Delta(G) - \Delta(H)$.

Remark 7.40. *Some properties of semisimple para-Kähler coset spaces.*

(1) The space $G/C(Z)$ is the coadjoint orbit of G through f, and so it is a Hamiltonian G-space in the sense of Kostant [140].

(2) A semisimple para-Kähler coset space of the ν-th kind is a para-Hermitian symmetric space if and only if $\nu = 1$.

The semisimple para-Kähler coset spaces enjoy some properties similar to those of semisimple symmetric para-Hermitian spaces:

(3) A semisimple para-Kähler coset space $M = G/C(Z)$ is diffeomorphic to the cotangent bundle of a certain R-space. If G is complex semisimple, then $G/C(Z)$ is holomorphically equivalent to the cotangent bundle of a certain Kähler space.

(4) A semisimple para-Kähler coset space $M = G/C(Z)$ is equivariantly imbedded in a certain space $\tilde{M}$ as the G-orbit through a certain point under the diagonal G-action. The image of M is open and dense in $\tilde{M}$. In particular, $\tilde{M}$ is viewed as a G-equivariant compactification of M. If G is complex semisimple, then the above imbedding is holomorphic.

We recall here that the infinitesimal classification of semisimple para-Kähler coset spaces of the second kind has been given in [123, 125].

8 Para-Kähler space forms

8.1 Para-Kähler manifolds of constant paraholomorphic sectional curvature

Consider the tensor field R' on the para-Kähler manifold (M, g, J) defined by

$$R'(X,Y,Z,W) = \frac{1}{4}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) - g(X,JZ)g(Y,JW) \\ + g(X,JW)g(Y,JZ) - 2g(X,JY)g(Z,JW)\}\,, \; X,Y,Z,W \in \mathfrak{X}(M).$$

This tensor field was independently defined in [188] and [87]. We recall from Section 4.3 that $H(X)$ denotes the paraholomorphic sectional curvature defined by a vector X. We have the following:

Theorem 8.1. ([87]) *Let (M, g, J) be a para-Kähler manifold such that for each $x \in M$, there exists $c_x \in \mathbb{R}$ satisfying $H(X) = c_x$ for every $X \in T_xM$ such that $g(X,X)g(JX,JX) \neq 0$. Then the Riemann-Christoffel tensor R satisfies $R = cR'$, where c is the function defined by $x \mapsto c_x$. And conversely.*

Definition 8.2. A para-Kähler manifold (M, g, J) is said to be of *constant paraholomorphic sectional curvature* c if it satisfies the conditions of the previous theorem.

One has the following Schur-type result:

Theorem 8.3. ([87]) *Let (M, g, J) be a para-Kähler manifold with constant paraholomorphic sectional curvature c. If* $\dim M > 2$, *then c is a constant function.*

Consequently, if $\dim M = 2$, in order to guarantee that a para-Kähler manifold has constant paraholomorphic sectional curvature c, one must assume that c is a constant function.

On the other hand, we have the following result concerning the *ordinary* sectional curvature:

Theorem 8.4. ([87]) *Let (M, g, J) be a para-Kähler manifold with constant paraholomorphic sectional curvature c. Then the* (ordinary) *sectional curvature of the planes of TM is:*

(1) *equal to c, if* $\dim M = 2$.

(2) *unbounded if* $\dim M > 2$, $c \neq 0$.

As is well known, É. Cartan [41] defined for a Riemannian manifold the axiom of the plane and the axiom of free mobility, and proved the equivalence of the constant curvature's property with each axiom, and also with the existence of a geodesic representation in the ordinary space. The almost complex and Kähler analogs are also well known ([116, 244, 261]). For a para-Kähler manifold, the concepts of paraholomorphic free mobility and the axiom of paraholomorphic planes – for several axiom of planes related to different types of sections see [104] – are given in [198]. In that paper, the following result is given:

Theorem 8.5. *Let (M, g, J) be a para-Kähler manifold with* $\dim M > 2$. *Then the following properties are equivalent:*

(1) *M is a space of constant paraholomorphic sectional curvature c.*

(2) *The Riemann-Christoffel curvature tensor R has the expression*

$$R(X,Y,Z,W) = \frac{c}{4}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W) + g(X,JZ)g(JY,W) \\ - g(Y,JZ)g(JX,W) + 2g(X,JY)g(JZ,W)\}, \quad X,Y,Z,W \in \mathfrak{X}(M).$$

(3) *M admits paraholomorphic free mobility.*

(4) *M is paraholomorphically projectively flat.*

(5) *M satisfies the axiom of paraholomorphic planes.*

Theorem 8.6. ([188, 189])

(1) *A para-Kähler manifold M with* $\dim M > 4$ *admits a paraholomorphically projective transformation onto a para-Kähler space of constant paraholomorphic sectional curvature if and only if M is a space of constant paraholomorphic sectional curvature.*

(2) *There is a paraholomorphically projective transformation of a para-Kähler space M onto a locally symmetric space if and only if M is of constant paraholomorphic sectional curvature.*

Some results on manifolds with constant nonvanishing paraholomorphic sectional curvature are given in [25] and [104]. We quote from the last reference the following result:

Theorem 8.7. *Let M be a para-Kähler manifold. If the paraholomorphic sectional curvatures in every point are bounded, i.e., for an arbitrary nondegenerate paraholomorphic section α in $T_p(M)$, we have $|K(\alpha,p)| \leq c(p)$, then M is of constant paraholomorphic sectional curvature.*

The general expression of the metric and the almost product structure of para-Kähler space forms in normal coordinates is given in [87]. From that expression, the following result is obtained in [88]:

Theorem 8.8. *Any C^∞ para-Kähler manifold M^{2n} of constant paraholomorphic sectional curvature c is harmonic, with characteristic function*

$$\Delta\Omega = \begin{cases} 1+\sqrt{c\Omega/2}\left\{(2n-1)\cot\sqrt{c\Omega/2}-\tan\sqrt{c\Omega/2}\right\}, & c\Omega>0 \\ 1+\sqrt{-c\Omega/2}\left\{(2n-1)\coth\sqrt{-c\Omega/2}-\tanh\sqrt{-c\Omega/2}\right\}, & c\Omega<0 \\ n, & c=0. \end{cases}$$

As usual, Ω denotes here the distance function (see [230]) and Δ the Laplacian.
On the other hand, a formula for the Laplacian of para-Kähler space forms is given in [232].
We quote here the following result:

Theorem 8.9. ([193]) *A para-Kähler manifold is H-concircularly flat if it is of constant paraholomorphic sectional curvature.*

Some conditions for a para-Kähler space manifold – the elliptic and parabolic cases are also considered – to be a para-Kähler space form, in terms of HP-transformations (see Subsection 6.2) can be found in [60].

8.2 The paracomplex projective models $P_n(\mathbb{B})$

As we have said in Section 2.5, Rozenfeld-Libermann's paracomplex projective spaces $P_n(\mathbb{A})$ were introduced in [144] and [223], [224, p.578]. They are endowed in [144] with an almost para-Hermitian structure (g, J), with the wish to make them the models of paraholomorphic sectional curvature, by paralleling the construction in the complex case.

We recall that in the complex case one can– see, for instance, [42] – normalize the complex homogeneous coordinate vectors Z, putting $Z_0 = Z/(Z,Z)^{1/2}$ and thus give to the Fubini-Study metric on the complex projective space $P_n(\mathbb{C})$ the expression $g = (dZ_0, dZ_0) - (dZ_0, Z_0)(Z_0, dZ_0)$. In [144, p. 89], the author uses a similar procedure and obtains a pseudo-Riemannian metric of "Fubini-Study" type $g = (de_0, de_0') - (e_0, de_0')(e_0', de_0)$. This expression is valid only locally, in the open subset of the space $P_n(\mathbb{A}) \approx P_n(\mathbb{R}) \times P_n(\mathbb{R})$ complementary of a singular hyperquadric.

The paracomplex projective models $P_n(\mathbb{B})$ were introduced in [87]. They are diffeomorphic neither to our paracomplex projective space nor to Libermann's paracomplex projective space (see Section 2.5), but they are, for $n > 1$, models of para-Kähler manifolds of nonvanishing constant paraholomorphic curvature, as is proved in [87]. Notice that, since the metric has signature (n, n), in order to change the sign of the sectional curvature it suffices changing the sign of the metric. Moreover, as they are a projective spaces of a certain kind, we shall call them the *paracomplex projective models.*

We recall here its definition and first properties. Let $\mathbb{B}$ be the vector space $\mathbb{R}^2$ with the product $(a,b)(a',b') = (aa',bb')$, with which $\mathbb{B}$ is a commutative $\mathbb{R}$-algebra via the inclusion $a \mapsto (a,a)$, $a \in \mathbb{R}$. If we define the conjugate $\bar{w}$ of an element $w = (a,b) \in \mathbb{B}$ by $\bar{w} = (b,a)$, then an element

w is real if $\bar{w} = w$ and it is invertible if $w\bar{w} \neq 0$. We put $\mathbb{B}_+ = \{(a,b) \in \mathbb{B} : a > 0, b > 0\}$. Then $\mathbb{B}_+$ is a Lie group. Let $\mathbb{B}_0^{n+1} = \{z = (z^\alpha) \in \mathbb{B}^{n+1} : \langle z, z\rangle > 0\}$, where $\langle z, z\rangle = \sum_{\alpha=0}^{n} z^\alpha \bar{z}^\alpha$.

We denote by $\mathfrak{gl}(n+1,\mathbb{B})$ the algebra of $(n+1)\times(n+1)$-matrices with elements in $\mathbb{B}$. Then $\mathfrak{gl}(n+1,\mathbb{B}) = \mathfrak{gl}(n+1,\mathbb{R}) \times \mathfrak{gl}(n+1,\mathbb{R})$. We have the Lie group

$$U(n+1,\mathbb{B}) = \{Z \in \mathfrak{gl}(n+1,\mathbb{B}) : \langle Zz, \bar{Z}\bar{z}\rangle = \langle z, \bar{z}\rangle, \quad z \in \mathbb{B}^{n+1}\}.$$

Let $P_n(\mathbb{B})$ be the quotient space of $\mathbb{B}_0^{n+1}$ under the equivalence relation given by $(z^\alpha) \sim (qz^\alpha)$, $q \in \mathbb{B}_+$. Then if $\pi\colon \mathbb{B}_0^{n+1} \longrightarrow P_n(\mathbb{B})$ is the natural projection, we can identify $\pi(z)$ with the unique element $w = qz$ such that $\langle w, \bar{w}\rangle = 1$, $\langle w, w\rangle = \langle \bar{w}, \bar{w}\rangle$, where $q = (a,b) \in \mathbb{B}_+$. Thus,

$$P_n(\mathbb{B}) \approx \{(u,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \langle u, v\rangle = 1, \ \langle u, u\rangle = \langle v, v\rangle\}.$$

Since $Z(qz) = qZ(z)$ for all $Z \in U(n+1,\mathbb{B})$, $z \in \mathbb{B}_0^{n+1}$, $q \in \mathbb{B}_+$, the action of $U(n+1,\mathbb{B})$ passes to the quotient $P_n(\mathbb{B})$.

Some geometric properties of the paracomplex projective model and its reduced space $P_n(\mathbb{B})/\mathbb{Z}_2$ are studied in [88]. A geometric realization of the space $P_1(\mathbb{B})/\mathbb{Z}_2$ is also considered there (see also [120]).

In [89, 90], it is shown that *the spaces $P_n(\mathbb{B})$ must be considered as one of the most natural homogeneous pseudo-Riemannian spaces.* In fact, it is shown that in order to obtain the geometry of these spaces, it suffices only to give a real finite dimensional vector space and its dual space, which will be now explained.

Let E be a real $(n+1)$-dimensional vector space, and E^* its dual space. On the space $E \oplus E^*$ there exist:

(1) A natural non-degenerate bilinear form $\langle\, ,\, \rangle$ given by

$$\langle x + \alpha, y + \beta\rangle = (2/c)(\alpha(y) + \beta(x)), \qquad x, y \in E, \quad \alpha, \beta \in E^*, \quad 0 \neq c \in \mathbb{R}.$$

(2) A (1,1) tensor J_0 such that $J_0|_E = id_E$, $\quad J_0|_{E^*} = -id_{E^*}$.

The subgroup of $GL(E \oplus E^*)$ which preserves $\langle\, ,\, \rangle$ and J_0 can be identified with $GL(E)$. We introduce in

$$(E \oplus E^*)_+ = \{x + \alpha \in E \oplus E^* : \langle x + \alpha, x + \alpha\rangle = (4/c)\alpha(x) > 0\}$$

the following equivalence relation $\sim$: $x + \alpha \sim ax + b\alpha$, $a > 0$, $b > 0$, and define

$$P(E \oplus E^*) = (E \oplus E^*)_+/\sim .$$

The identity component $GL_0(E)$ of $GL(E)$ acts transitively on the pseudosphere $S = \{x + \alpha \in E \oplus E^* : \alpha(x) = 1\}$ and also on $P(E \oplus E^*)$, making it a homogeneous manifold $(P(E \oplus E^*), GL_0(E))$ and the base space of a fibre bundle $p\colon S \longrightarrow P(E \oplus E^*)$ with fibre $\mathbb{R}$ and such that the subgroup $\{aI \in GL_0(E) : a > 0\}$ of $GL_0(E)$ acts transitively on the fibres. From this bundle we can endow $P(E \oplus E^*)$ with a pseudo-Riemannian metric $\langle\, ,\, \rangle$ and an almost product structure J induced very simply via S from the structures in $E \oplus E^*$, which, as it is proved, make $(P(E \oplus E^*), GL_0(E))$ a para-Kähler space form *isomorphic to the paracomplex projective model* $P_n(\mathbb{B})$. Moreover, the construction of $(P(E \oplus E^*), \langle\, ,\, \rangle, J)$ is natural with respect to the category of finite dimensional real vector spaces. In this sense, because of the economy of the initial data, the geometry of these spaces have a right to stand immediately after affine and projective geometry and prior to, say, the geometry of the sphere. That space is thus one of the more natural pseudo-Riemannian homogeneous spaces.

8.3 Classification of para-Kähler space forms

The classification of para-Kähler space forms relies on the two following results:

Theorem 8.10. ([87]) *Any two complete, connected and simply connected para-Kähler manifolds of constant and equal paraholomorphic sectional curvature c are paraholomorphically isometric (we assume that c is a constant function).*

Theorem 8.11. ([87, 88, 94])

(1) *The space $(P_n(\mathbb{B}), g, J)$, where g is the metric*

$$g = \frac{2}{c(1+\langle x, y\rangle)}(dx_i \otimes dy_i + dy_i \otimes dx_i - \frac{x_i y_j}{1+\langle x, y\rangle}(dy_i \otimes dx_j + dx_j \otimes dy_i)),$$

$0 \neq c \in \mathbb{R}$, and J the almost product structure

$$J = \frac{\partial}{\partial x_i} \otimes dx_i - \frac{\partial}{\partial y_i} \otimes dy_i$$

(both in the coordinates x_i, y_i given in [87]), is, for $n > 1$, the model of the $2n$-dimensional para-Kähler space forms of paraholomorphic sectional curvature $c \neq 0$.

(2) *The space $(\mathbb{R}^2, g, J)$, where g is the metric*

$$g = \frac{4}{c}(\cosh^2 2y\, dx \otimes dx - dy \otimes dy), \; 0 \neq c \in \mathbb{R}, \tag{8.1}$$

and J the almost product structure

$$J = -\frac{1}{\cosh 2y}\frac{\partial}{\partial x} \otimes dy - \cosh 2y \frac{\partial}{\partial y} \otimes dx, \tag{8.2}$$

both in the coordinates (x, y) of $\mathbb{R}^2$, is the model of the para-Kähler space forms of dimension 2 and paraholomorphic sectional curvature $c \neq 0$.

(3) *The space $(\mathbb{R}^{2n}_n, g, J)$, $n \geq 1$, where g is the metric and J the almost product structure given by*

$$g = dx_i \otimes dy_i + dy_i \otimes dx_i, \quad J = \frac{\partial}{\partial x_i} \otimes dx_i - \frac{\partial}{\partial y_i} \otimes dy_i,$$

(x_i, y_i) being the coordinates of $\mathbb{R}^{2n}$, is the model of the para-Kähler space forms of dimension $2n \geq 2$ and paraholomorphic sectional curvature $c = 0$.

The para-Kähler space forms are classified in [94, 95] for the cases included in the following results (see those papers for the details):

Theorem 8.12. *Let M^{2n}_n be a connected complete homogeneous para-Kähler manifold of constant paraholomorphic sectional curvature c and dimension $2n$. Then, M^{2n}_n is paraholomorphically isometric to a manifold of one of the following types:*

(1) *$P_n(\mathbb{B})/\Gamma$, element of $\mathcal{TS}$ for every $n > 1$, or $\mathcal{TZ}$ for odd $n > 1$, or $\mathcal{TD}$, $\mathcal{TT}$, $\mathcal{TO}$ or $\mathcal{TI}$ for $n + 1 \equiv 0 \pmod 4$ (if $c \neq 0$, $n > 1$);*

(2) *a Lorentzian covering of* $(P_1(\mathbb{B})/\mathbb{Z}_2, g, J)$*, where* g *is the metric* (8.1) *for* $x \in [0, \pi]$ *and* J *the almost product structure* (8.2) (*if* $c \neq 0,\ n = 1$);

(3) $\mathbb{R}^{2n}_n/\Gamma$*, where* $\mathbb{R}^{2n}_n$ *is endowed with the structure* (g, J) *in* (3) *in Theorem* 8.11*, and* Γ *is a group of pure translations* (*if* $c = 0,\ n \geq 1$).

Theorem 8.13. *Let* M^{2n}_n *be a complete connected para-Kähler manifold of constant paraholomorphic sectional curvature* $c \neq 0$ *and dimension* $2n > 2$*. Then it is paraholomorphically isometric to a manifold diffeomorphic to the tangent bundle of a spherical space form* S^n/Γ*; that is, to a manifold of the type* $T(S^n/\Gamma)$*, where* $\Gamma = (\sigma_1 \oplus \ldots \oplus \sigma_r)/G$*,* G *being a finite group and* $\sigma_1, \ldots, \sigma_r$ *fixed point free irreducible orthogonal representations such that* $\sum_i \deg \sigma_i = n + 1$.

The spaces in (1) in Theorem 8.12 are diffeomorphic to the tangent bundles of homogeneous spherical space forms (see [254]). This is the reason for the notation $T\mathcal{S}$, etc.; that is, one writes $\mathcal{S}$ for *spherical*, $\mathcal{Z}$ for *cyclic*, $\mathcal{D}$ for *dihedral*, $\mathcal{T}$ for *tetrahedral*, $\mathcal{O}$ for *octahedral* and $\mathcal{I}$ for *icosahedral*. As one can see, this phenomenon appears again in the non-homogeneous case. On the other hand, notice the difference with the Kähler case, where one has only the complex projective space $P_n(\mathbb{C})$, whose space forms are not Hermitian [254].

Theorem 8.14. ([95]) *Let* (M^{2n}, g, J) *be a connected, complete para-Kähler manifold. Then:*

(1) $\dim \mathrm{Aut}(M, g, J) \leq n(n + 2)$;

(2) $\dim \mathrm{Aut}(M, g, J) = n(n + 2)$ *if and only if* M *is paraholomorphically isometric to one of the following homogeneous para-Kähler space forms*:

(a) $P_n(\mathbb{B})$ *or* $P_n(\mathbb{B})/\mathbb{Z}_2$, $\quad n > 1,\ c \neq 0$.

(b) $\mathbb{R}^2$, $\quad c \neq 0$.

(c) $\mathbb{R}^{2n}_n$, $\quad c = 0$.

Remark 8.15. *Submanifolds of* $P_n(\mathbb{B})$. The geodesics of $P_n(\mathbb{B})$ are studied in [88]. The totally geodesic submanifolds in [89, 90]. The classification of the nondegenerate totally umbilical pseudo-Riemannian submanifolds can be found in [91].

9 Some open problems

(3.1) (1) To give the homotopy classification of paracomplex structures on (perhaps a restricted type of) even-dimensional manifolds (see [92]).

(2) Does there exist a paracomplex structure on the pseudosphere S^6_3? (see [56, 113, 144]).

(3) To find "obstacles" – in terms of characteristic classes, for instance – for a manifold to have a paracomplex structure (see [70, 100], [144, p. 27], [155]).

(4.1) (1) To study and – mainly – characterize the almost para-Hermitian manifolds whose Riemann-Christoffel curvature tensor R satisfies the condition which one has in the Kähler case: $R(X, Y, JZ, W) + R(X, Y, Z, JW) = 0$. (See Proposition 5.3.)

(2) Classify the non-para-Kähler almost para-Hermitian manifolds of constant paraholomorphic sectional curvature.

(4.3) (1) To give examples of each one of the 136 classes of almost para-Hermitian manifolds (see [93]). One of the possible ways of to obtain results is to consider the general structures on the tangent and cotangent bundle in [48, 3.5] and [49, 50, 51]– where examples of the primitive classes are given – and the large series of properties given in the papers by Rosca and others.

(2) To characterize each of the 136 classes, and relate these characterizations to several authors' results (for instance, [247], see Subsection 4.3).

(3) To prove that the inclusion relations among the 136 classes are strict.

(4.9) To characterize the non-reductive homogeneous para-Hermitian manifolds (see [101]).

(5.1) (1) Is there a gap in the possible dimensions of the automorphism group of an almost para-Hermitian manifold? ([95]).

(2) Under what conditions is an (almost) para-Hermitian or para-Kähler manifold a tangent or cotangent manifold?

(3) How large is – in the appearing signatures – the family of pseudo-Riemannian flat space forms given via Cruceanu's construction from para-Kähler manifolds?

(5.4) A specific problem: How to construct a Frenet field of frames along a null curve of a para-Kähler manifold?

(5.6) To give an example of a Bochner flat para-Kähler space form with constant scalar curvature and nonvanishing Pontrjagin classes (see [23]).

(7.2) To study and classify the para-Hermitian symmetric spaces with non-semisimple group.

(7.3) (1) To give the explicit expression of the para-Kähler structure of all the para-Hermitian symmetric spaces with simple group (see [127, 96]). The first cases to be studied could be the quaternionic para-Grassmannians; that is, the para-Hermitian symmetric spaces corresponding to the symmetric pair $(\mathfrak{su}^*(2m+2n), \mathfrak{su}^*(2m)+\mathfrak{su}^*(2n)+\mathbb{R})$, which are spaces diffeomorphic to the cotangent bundles of the quaternionic Grassmannians $G_{m.n}(\mathbb{H})$.

(2) To study the para-Hermitian symmetric spaces with simple group – different to the well-known case $T^*(O(n))$ – as (up to diffeomorphism) phase spaces of dynamical systems.

(8.2) To study the reflector bundles of the paracomplex projective models $P_n(\mathbb{B})$ and of their reduced spaces $P_n(\mathbb{B})/\mathbb{Z}_2$.

References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Cummings, 1978.

[2] N.N. Adamushko, *The geometry of simple and quasisimple Lie groups of class G_2*, Moskow Oblast. Ped. Inst. Ucen. Zap. **253** (1969) 23–42 (Russian).

[3] F.S. Ahrarov, *Analytical surfaces with constant curvature in the spaces of* 0*-couples*, Izv. Mat. **5** (1978), 127–130 (Russian).

[4] F.S. Ahrarov, *On normalized surfaces in the space of* 0*-couples*, Trudy Geom. Sem. **10** (1978) 18–36 (Russian).

[5] F.S. Ahrarov and A.P. Norden, *Intrinsic geometry of analytical surfaces in the space of non-degenerate* 0*-couples*, Izv. Mat. **8** (1978) 19–30 (Russian).

[6] C. Allamigeon, *Espaces homogènes symétriques à groupe semi-simple*, C. R. Acad. Sci. Paris Sér. I Mat. **243** (1956) 121–123.

[7] F. Amato, *Sur une classe de variétés paracomplexes possédant la propriété concirculaire*, Riv. Mat. Univ. Parma (4) **8** (1982) 223–234.

[8] F. Amato, *CR-sous-variétés d'une variété parakählerienne possédant la propriété de Poisson*, Rend. Mat. Appl. (7) **4** (1984) 351–355.

[9] F. Amato, *CR-sous-variétés co-isotropes inducées dans une C^∞-variété pseudo-riemannienne neutre*, Boll. Un. Mat. Ital. (7) **4** (1985) 433–440.

[10] F. Amato, *Immersions impropres dans une variété paracomplexe possédant la propriété concirculaire*, Reports Math. Sem., Brescia, **9** (1988) 55–63.

[11] W. Ambrose and I.M. Singer, *On homogeneous manifolds*, Duke Math. J. **25** (1958) 647–669.

[12] G. Arca, *Variétés pseudo-riemanniennes structurées par une connexion spin-euclidienne et possédant la propiété de Killing*, C. R. Acad. Sci. Paris Sér. I Mat. **290** (1980) 839–842.

[13] G. Arca, *Surfaces hyperboliques ayant la propriété géodésique*, Rend. Sem. Fac. Sci. Univ. Cagliari **49** (1979) 565–569.

[14] G. Arca, *$\mathfrak{A}$-conformal connections on a neutral semi-Riemannian manifold*, Tensor (N.S.) **47** (1988) 260–271.

[15] G. Arca, *On a class of paracomplex manifolds with geodesic connection*, Fachber. Math. und Informatik, Fernuniv. Aagen **37** (1990) 22–32.

[16] G. Arca, R. Caddeo and R. Rosca, *Variétés para-kähleriennes possédant la propriété concirculaire*, C. R. Acad. Sci. Paris Sér. I Mat. **286** (1978) 1209–1212.

[17] G. Arca and R. Rosca, *Variétés parakähleriennes possédant la propriété de Poisson*, Riv. Mat. Univ. Parma (4) **6** (1980) 281–286.

[18] C.P. Awasthi, *On almost hyperbolic spaces*, Publ. Inst. Math. (Beograd) (N.S.)**20** (1976) 41–49.

[19] C.P. Awasthi, *Some problems on hyperbolic Kähler recurrent space*, Nepali Math. Sci. Rep. **8** (1983) 29–36.

[20] C. Bejan, *A classification of the almost parahermitian manifolds*, Proc. Conference on Diff. Geom. and Appl., Dubrovnik, 1988, 23–27.

[21] C. Bejan, *Almost parahermitian structures on the tangent bundle of an almost paracohermitian manifold*, Proc. Fifth Nat. Sem. Finsler and Lagrange spaces, Brasov, 1988, 105–109.

[22] C. Bejan, *CR-submanifolds of hyperbolical almost Hermitian manifolds*, Demonstratio Math. **23** (1990) 335–343. Univ. "Al. I. Cuza", Iaşi, 1989, 27–30.

[23] C. Bejan, *The Bochner curvature tensor of a hyperbolic Kähler manifold*, Coll. Diff. Geom., Eger, Hungary, 1989, North-Holland, 93–99.

[24] C. Bejan, *Some examples of manifolds with hyperbolic structures* **14** (1994) 557–565.

[25] C. Bejan, *Structuri hiperbolice pe diverse spatii fibrate*, Ph. D. Thesis, Iaşi, 1990.

[26] C. Bejan, *The existence problem of hyperbolic structures on vector bundles*, Publ. Inst. Math. (Beograd) (N.S.) **53** (**67**) (1993) 133–138.

[27] A. Bejancu, *The transversal vector bundle of a degenerate hypersurface of an almost para-Hermitian manifold* (preprint).

[28] A. Bejancu and F. Etayo, *Degenerate hypersurfaces of almost-parahermitian manifolds*, J. Tensor Soc. India **12** (1994) 39–56.

[29] A. Bejancu and T. Otsuki, *General Finsler connections on a Finsler vector bundle*, Kōdai Math. J. **10** (1987) 143–152.

[30] M. Berger, *Les espaces symétriques non compacts*, Bull. Soc. Math. France **74** (1957) 85–177.

[31] F. Bien, *D-modules and spherical representations*, Math. notes, n. 39, Princeton Univ. Press, 1990.

[32] I. Bouzon, *Structures presque cohermitiennes*, C. R. Acad. Sci. Paris Sér. I Mat. **258** (1964) 412–415.

[33] K. Buchner and R. Rosca, *Variétés para-cokähleriennes à champ concirculaire horizontal*, C. R. Acad. Sci. Paris Sér. I Mat. **285** (1977) 723–726.

[34] K. Buchner and R. Rosca, *Co-isotropic submanifolds of a para-co-Kählerian manifold with concircular structure vector field*, J. Geom. **25** (1985) 164–177.

[35] K. Buchner and R. Rosca, *Invariant submanifolds and proper CR foliations on a paracoKählerian manifold with concircular structure vector field*, Rend. Circ. Mat. Palermo (2) **37** (1988), 161–173.

[36] M.T. Calapso and R. Rosca, *On para-Kählerian manifolds with conformal self-orthogonal connection*, Tensor (N.S.) **34** (1980) 235–241.

[37] M.T. Calapso and R. Rosca, *Cosymplectic quasi-Sasakian pseudo-Riemannian manifolds and coisotropic foliations*, Rend. Circ. Mat. Palermo (2) **36** (1987) 407–422.

[38] M. Capursi and A. Palombella, *On the almost hermitian structures associated to a generalized Lagrange space*, Mem. Sect. Ştiinţ. Acad. Romania Ser. IV, t. VIII, n. 1, 1983.

[39] M. Capursi and A. Palombella, *On the almost Hermitian models of a generalized Lagrange space*, Proc. 4th Nat. Sem. Finsler and Lagrange spaces, Brasov, 1986, 115–128. Soc. Şti. Mat. R. S. Rom., Bucharest, 1986.

[40] É. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54** (1926) 214–264.

[41] É. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Gauthier-Villars, 1963.

[42] S.S. Chern, *Einstein hypersurfaces in a Kählerian manifold of constant holomorphic curvature*, J. Differential Geom. **1** (1967) 21–31.

[43] W.K. Clifford, *A preliminary sketch on biquaternions*, Proc. London Math. Soc. **4** (1873) 381–395. Reprinted in: *Mathematical papers*, Chelsea Publ., New York, 1968, 181–200.

[44] W.K. Clifford, *On the theory of screws in a space of constant curvature*, id., 402–405.

[45] W.K. Clifford, *On the motion of a body in an elliptic space*, 1874, id., 378–384.

[46] W.K. Clifford, *Further note on biquaternions*, id., 385–396.

[47] V. Cruceanu, *Connexions compatibles avec certaines structures sur un fibré vectoriel banachique*, Czechoslovak Math. J. **24** (1974) 126–142.

[48] V. Cruceanu, *Structures et connexions classiques sur une variété différentiable*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. a Mat. (N. S.) **22** (1976) 181–190.

[49] V. Cruceanu, *Certaines structures sur le fibré tangent*, Proc. Inst. Math. Iaşi, Ed. Acad. Rom., 1976, 41–49.

[50] V. Cruceanu, *Une structure parakählerienne sur le fibré tangent*, Tensor (N.S.) **39** (1982) 81–84.

[51] V. Cruceanu, *Sur certains morphismes des structures géométriques*, Rend. Mat. Appl. (7) **6** (1986) 321–332.

[52] V. Cruceanu, *Une classe de structures géométriques sur le fibré cotangent*, Tensor (N. S.) (to appear).

[53] V. Cruceanu, P. Fortuny and P.M. Gadea, *A survey on Paracomplex Geometry* Rocky Mountain J. Math. **26** (1995) 83–115.

[54] A. Crumeyrolle, *Variétés différentiables à coordonnées hypercomplexes. Application à une géométrisation et à une généralisation de la théorie d'Einstein-Schrödinger*, Ann. Fac. Sci. Univ. Toulouse **26** (1962) 105–137.

[55] A. Crumeyrolle, *Variétés différentiables à structure complexe hyperbolique. Applications à la théorie unitaire relativiste des champs*, Riv. Mat. Univ. Parma (4) **8** (1967) 27–53.

[56] B. Datta, $(1,2)$*-symplectic structures, nearly Kähler structures and* S^6, Internat. Centre for Theoret. Phys., IC/90/398, Miramare-Trieste, 1990.

[57] J.E. D'Atri and H.K. Nickerson, *The existence of special orthonormal frames*, J. Differential Geom. **2** (1968) 393–409.

[58] S. Deng and S. Kaneyuki, *An example of nonsymmetric dipolarizations in a Lie algebra*, Tokyo J. Math. **16** (1993) 509–511.

[59] F. Dillen and L. Verstraelen (Editors) *Differential Geometry in honor of Radu Rosca*, Kathol. Univ. Leuven, Dep. Wiskunde, 1991.

[60] G.D. Djelepov, *A note on the existence of HP-transformations*, C. R. Acad. Bulgare Sci. **37** (1984) 291–292.

[61] H. Doi, *A classification of certain symmetric Lie algebras*, Hiroshima Math. J. **11** (1981) 173–180.

[62] S. Donato, *Para-Hermitian manifold with exterior recurrent line element splitting*, Atti Soc. Peloritana **30** (1984) 17–20.

[63] S. Donato, *Neutral manifold* $M(\mathfrak{A}, g, W)$ *structured by a parallel conformal connection*, Accad. Pelor. dei Peric. **62** (1984) 137–147.

[64] S. Donato, *CICR submanifolds of a para-Kähler manifold having the self-orthogonal Killing property*, Accad. Pelor. dei Peric. **66** (1988) 283–292.

[65] K.K. Dube, *On almost hyperbolic Hermitian manifolds*, Anal. Univ. Timişoara, Ser. Şti. Mat. **11** (1973) 47–54.

[66] K.K. Dube and M.M. Misra, *Hypersurfaces immersed in an almost hyperbolic Hermite manifold*, C. R. Acad. Bulgare Sci. **34** (1981) 1343–1345.

[67] M.A. Dzavadov, *A realization of stratified space*, Trudy Sem. Vekt. Tenz. Anal. **8** (1950) 182–192 (Russian).

[68] M.A. Dzavadov, *Conformal transformations in Euclidean and pseudo-Euclidean spaces of an arbitrary number of dimensions as linear fractional transformations*, Dokl. Akad. Nauk. SSSR **86** (1952) 653–656 (Russian).

[69] M.A. Dzavadov, *Projective and non-Euclidean geometries over matrices*, Dokl. Akad. Nauk. SSSR **97** (1954) 769–772 (Russian).

[70] C. Ehresmann, *Sur les variétés presque complexes*, Act. Proc. Intern. Cong. Math., 1950, 412–419.

[71] H. Endo, *Submanifolds in contact metric manifolds and in almost cosymplectic manifolds*, Ph. D. thesis, Univ. "Al. I. Cuza", Iaşi, 1993.

[72] S. Erdem, *Harmonic maps as solutions of the wave equation, quadratic differentials and paraholomorphicity* (preprint).

[73] F. Etayo, *The paraquaternionic projective spaces*, Rend. Mat. Appl. (7) **13** (1993) 125–131.

[74] F. Etayo, *The structure tensors and the sectional curvature function of para-Kählerian space forms* (to appear in Rend. Sem. Mat. Messina).

[75] F. Etayo and M. Fioravanti, *CR-submanifolds of the paracomplex projective space* (to appear in Publ. Math. Debrecen).

[76] F. Etayo and M. Fioravanti, *Classification of submanifolds and CR-submanifolds of an almost para-Hermitian manifold of dimension* 4 (preprint).

[77] F. Etayo, M. Fioravanti and U.R. Trías, *On the submanifolds of an almost para-Hermitian manifold* (preprint).

[78] F. Etayo and P.M. Gadea, *Paraholomorphically projective vector fields*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. a Mat. (N. S.) **38** (1992) 201–210.

[79] F. Etayo and R. Rosca, *On para-Kaehlerian manifolds having the skew-symmetric property* (to appear in An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. a Mat. (N. S.)).

[80] F. Etayo and U.R. Trías, *Paraholomorphic bisectional curvature* (preprint).

[81] H. Farran, *Almost product Riemannian manifolds*, Czechoslovak Math. J. **33** (1983) 119–125.

[82] H. Farran and M.S. Zanoun, *On hyperbolic Hermite manifolds*, Publ. Inst. Math. (Beograd) **46** (1989) 173–182.

[83] M. Flensted-Jensen, *Spherical functions on a real semisimple group. A method of reduction to the complex case*, J. Funct. Anal. **30** (1978) 106–146.

[84] M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Ann. of Math. (2) **111** (1980) 253–311.

[85] H. Freudenthal, *Lie groups in the foundations of geometry*, Adv. Math. **1** (1970) 145–190.

[86] A. Fujimoto, *Theory of G-structures*, Study Group of Geom., 1972.

[87] P.M. Gadea and A. Montesinos Amilibia, *Spaces of constant paraholomorphic sectional curvature*, Pacific J. Math. **136** (1989) 85–101.

[88] P.M. Gadea and A. Montesinos Amilibia, *Some geometric properties of parakählerian space forms*, Rend. Sem. Fac. Sci. Univ. Cagliari **59** (1989) 131–145.

[89] P.M. Gadea and A. Montesinos Amilibia, *The paracomplex projective spaces as symmetric and natural spaces*, Indian J. Pure Appl. Math. **23** (1992) 261–275.

[90] P.M. Gadea and A. Montesinos Amilibia, *The paracomplex projective model and the para-Grassmannian manifolds*, in "Contribuciones matemáticas. Estudios en honor del Profesor J.J. Etayo", Univ. Complutense, Madrid, 1994, 315–334.

[91] P.M. Gadea and A. Montesinos Amilibia, *Totally umbilical pseudo-Riemannian submanifolds of the paracomplex projective space*, Czechoslovak Math. J. **44** (1994) 741-756.

[92] P.M. Gadea and J. Muñoz Masqué, *Homotopy classification of f-structures on orientable* 3-*manifolds*, Geom. Dedicata **31** (1989) 199–205.

[93] P.M. Gadea and J. Muñoz Masqué, *Classification of almost para-Hermitian manifolds*, Rend. Mat. Appl. (7) **11** (1991) 377–396.

[94] P.M. Gadea and J. Muñoz Masqué, *Classification of homogeneous parakählerian space forms*, Nova J. Algebra Geom. **1** (1992) 111–124.

[95] P.M. Gadea and J. Muñoz Masqué, *Classification of nonflat parakählerian space forms*, Houston J. Math. **21** (1995) 89–94.

[96] P.M. Gadea and J. Muñoz Masqué, *Nonflat pseudo-Riemannian space forms and homogeneous pseudo-Riemannian structures of class* $\mathcal{S}_1$, Publ. Math. Debrecen **47** (1995) 167–72.

[97] P.M. Gadea and J. Muñoz Masqué, *Symmetric structures on the cotangent bundles of the real and complex Grassmannians*, Indian J. Pure Appl. Math. **27** (1996) 1–11.

[98] P.M. Gadea and J. Muñoz Masqué, *A-differentiability and A-analyticity*, Proc. Amer. Math. Soc. **124** (1996) 1437–43.

[99] P.M. Gadea and J. Muñoz Masqué, $\mathbb{B}$-*Stein manifolds* (preprint).

[100] P.M. Gadea and J.A. Oubiña, *Existence of* $J(4,2)$-*structures*, Riv. Mat. Univ. Parma (4) **9** (1983) 179–184.

[101] P.M. Gadea and J.A. Oubiña, *Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures*, Houston J. Math. **18** (1992) 449–465.

[102] P.M. Gadea and J.A. Oubiña, *Homogeneous almost para-Hermitian structures*, Indian J. Pure Appl. Math. **26** (1995) 351–362.

[103] P.M. Gadea and J.A. Oubiña, *Reductive homogeneous pseudo-Riemannian manifolds*, Monats. Math. **124** (1997) 17–34.

[104] G. Ganchev and A. Borisov, *Isotropic sections and curvature properties of hyperbolic Kaehlerian manifolds*, Publ. Inst. Math. (Beograd) (N.S.) **38** (1985) 183–192.

[105] V.V. Goldberg and R. Rosca, *Biconformal vector fields on manifolds endowed with a certain differential conformal structure*, Houston J. Math. **14** (1988) 81–95.

[106] J.T. Graves, *On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables*, Phil. Magaz., London-Edinburgh-Dublin, **26** (1845) 315–320.

[107] A. Gray and L.M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. di Mat. **123** (1980) 35–58.

[108] I. Grifone, *Structures presque γ-complexes*, Thèse 3ème cycle, Grenoble, 1965.

[109] J.M. Hernando, P.M. Gadea and A. Montesinos Amilibia, *G-structures defined by a tensor field of electromagnetic type*, Rend. Circ. Mat. Palermo (2) **34** (1985) 202–218.

[110] J. Hilgert, *The hyperboloid as ordered symmetric space*, Sem. Sophus Lie **1** (1991) 135–142.

[111] J. Hilgert, G. 'Olafsson and B. Ørsted, *Hardy spaces associated to symmetric spaces of Hermitian type*, Math. Gött., Heft 29, 1989.

[112] Z.-X. Hou, *On homogeneous paracomplex and para-Kähler manifolds*, Chinese J. Contemp. Math. **15** (1994), 193–206.

[113] C.C. Hsiung, *Nonexistence of a complex structure on the six-sphere*, Bull. Inst. Math. Acad. Sinica **14** (1986) 231–247.

[114] S. Ianuş and C. Udriste, *Asupra spatiului fibrat tangent al unei varietàti diferentiabile*, Stu. şi Cerc. Mat. **22** (1970) 599–611.

[115] S. Ianuş and R. Rosca, *Variétés para-Kähleriennes structurées par une connexion géodesique*, C. R. Acad. Sci. Paris Sér. I Mat. **280** (1975) 1621–1623.

[116] S. Ishihara, *On holomorphic planes*, Ann. Mat. Pura Appl. (4) **47** (1959) 197–241.

[117] G. Jensen and M. Rigoli, *Neutral surfaces in neutral four-spaces*, Matematiche (Catania) **45** (1991) 407–443.

[118] E. Kähler, *Über eine bemerkenswerte Hermitesche Metrik*, Abh. Math. Sem. Univ. Hamburg **9** (1933) 173–186.

[119] M. Kanai, *Geodesic flows of negatively curved manifolds with stable and unstable flows*, Ergodic Theory Dynamical Systems **8** (1988) 215–239.

[120] S. Kaneyuki, *On classification of parahermitian symmetric spaces*, Tokyo J. Math. **8** (1985) 473–482.

[121] S. Kaneyuki, *On orbit structure of compactifications of parahermitian symmetric spaces*, Japan. J. Math. (N.S.) **13** (1987) 333–370.

[122] S. Kaneyuki, *A decomposition theorem for simple Lie groups associated with parahermitian symmetric spaces*, Tokyo J. Math. **10** (1987) 363–373.

[123] S. Kaneyuki, *On a remarkable class of homogeneous symplectic manifolds*, Proc. Japan Acad. Ser. A Math. Sci. **67** (1991) 128–131.

[124] S. Kaneyuki, *Homogeneous symplectic manifolds and dipolarizations in Lie algebras*, Tokyo J. Math. **15** (1992) 313–325.

[125] S. Kaneyuki, *On the subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_{ev}$ of semisimple graded Lie algebras*, J. Math. Soc. Japan **45** (1993) 1–19.

[126] S. Kaneyuki and H. Asano, *Graded Lie algebras and generalized Jordan triple systems*, Nagoya Math. J. **112** (1988), 81–115.

[127] S. Kaneyuki and M. Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math. **8** (1985) 81–98.

[128] S. Kaneyuki and M. Kozai, *On the isotropy group of the automorphism group of a parahermitian symmetric space*, Tokyo J. Math. **8** (1985) 483–490.

[129] S. Kaneyuki and F. Williams, *On a class of quantizable co-adjoint orbits*, Algebras Groups Geom. **2** (1985) 70–94.

[130] S. Kaneyuki and F. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. **99** (1985) 173–187.

[131] O. Kasabov, *Hyperbolic Kaehler manifold of constant holomorphic sectional curvature*, Annuar. Univ. Sofia, Fac. Math. Mech. **79** (1985) 143–147.

[132] P.F. Kelly and R.B. Mann, *Ghost properties of algebraically extended theories of gravitation*, Classical Quantum Gravity **3** (1986) 705–712.

[133] V.F. Kirichenko, *Tangent bundles from the point of view of generalized Hermitian geometry*, Izv. Vyssh. Uchebn. Zaved, Math. **7**(**266**) (1984) 50–58 (Russian); english translation: Sov. Math. **28** (1984) 63–74.

[134] A. Kirillov, *Unitary representations of nilpotent groups*, Russian Math. Surveys **17** (1962) 53–104.

[135] A. Koranyi and J.A. Wolf, *Realization of Hermitian symmetric spaces as generalized half-planes*, Ann. of Math. (2) **81** (1965) 265–288.

[136] F. Klein, *Vergleichende Betrachtungen über neuere geometrische Vorschungen*, A. Deichert, Erlangen, 1872.

[137] S. Kobayashi and T. Nagano, *Filtered Lie algebras and geometric structures, I*, J. Math. Mech. **13** (1964) 875–908.

[138] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Intersc. Publ., 1963 and 1969.

[139] V.G. Kopp, *Geometry of a neutral space*, Trudy Geom. Sem. Kazan Univ. **11** (1979) 56–63; **12** (1980), 45–49 (Russian).

[140] B. Kostant, *Quantization and unitary representations (part I: prequantization)*, Lect. notes in Math., n. 170, Springer-Verlag, 1970, 87–208.

[141] A.P. Kotelnikov, *Twist calculus and some of its applications to geometry and mechanics*, Kazan, 1895 (Russian).

[142] G. Kunstatter and J.W. Moffat, *Geometrical interpretation of a generalized theory of gravitation*, J. Math. Phys. **24** (1986) 886.

[143] P. Libermann, *Sur les structures presque paracomplexes*, C. R. Acad. Sci. Paris Sér. I Mat. **234** (1952) 2517–2519.

[144] P. Libermann, *Sur le problème d'équivalence de certaines structures infinitésimales*, Ann. Mat. Pura Appl. (4) **36** (1954) 27–120.

[145] P. Libermann, *Sur la classification des structures hermitiennes*, Col. Geom. Dif., Santiago de Compostela, 1978, 168–191.

[146] R.B. Mann, *New ghost-free extensions of General Relativity*, Classical Quantum Gravity **6** (1989) 41–57.

[147] G. Markov and M. Prvanović, *Some spaces one of whose πH-projective curvature tensors vanishes*, Tensor (N.S.) **46** (1987) 387–396.

[148] G. Markov and M. Prvanović, *π-holomorphically planar curves and π-holomorphically projective transformations*, Publ. Math. Debrecen **37** (1990) 273–284.

[149] S. Matsumoto, *Discrete series for an affine symmetric space*, Hiroshima Math. J. **11** (1981) 53–79.

[150] O. Mayer, *Géométrie biaxiale différentielle des courbes*, Bull. Math. Soc. Roum. Sci. **4** (1938) 1–4.

[151] O. Mayer, *Biaxiale Differentialgeometrie der Kurven und Regelflëchen*, Anal. Sci. Univ. Jassy **27** (1941) 327–410.

[152] O. Mayer, *Geometria biaxiala diferentiala a suprafeţelor*, Lucr. Ses. Şti. Acad. Rep. Pop. Romàne, 1950, 1–7.

[153] I. Mihai and C. Nicolau, *Almost product structures on the tangent bundle of an almost paracontact manifold*, Demonstratio Math. **15** (1982) 1045–1058.

[154] I. Mikesh and G.A. Starko, *On hyperbolically Sasakian and equidistant hyperbolically Kählerian spaces*, Ukran. Geom. Sbor. **32** (1989) 92–98 (Russian).

[155] J. Milnor, *Characteristic classes*, Princeton Univ. Press, 1974.

[156] R. Miron and G. Atanasiu, *Existence et arbitraire des connexions compatibles aux structures Riemann généralisées du type hermitien*, Tensor (N.S.) **38** (1982) 8–12.

[157] R. Miron and G. Atanasiu, *Hyperbolical almost Hermite structures*, Proc. Nat. Coll. Geom. Top., Busteni, 1981. Publ. Univ. Bucharest, 1983, 234–243.

[158] R. Miron and G. Atanasiu, *Existence et arbitrariété des connexions compatibles à une structure Riemann généralisée du type presque k-horsymplectique métrique*, Kōdai Math. J. **6** (1983) 228–237.

[159] J.W. Moffat, *New theory of gravitation*, Phys. Rev. D. (3) **19** (1979) 3554–3558.

[160] J.W. Moffat, *A solution of the Cauchy initial value problem in the nonsymmetric theory of gravitation*, J. Math. Phys. **21** (1980) 1798–1801.

[161] J.W. Moffat, *Higher-dimensional Riemannian geometry and quaternion and octonion spaces*, J. Math. Phys. **25** (1984) 347–350.

[162] J.W. Moffat, *Nonsymmetric gravitation theory and its experimental consequences*, Proc. Sir Arthur Eddington cent. Symp., vol. 1, Nagpur, 1984, World Sci. Publ., Singapore, 1984, 197–223.

[163] J.W. Moffat, *Chiral fermions in nonriemannian Kaluza-Klein theory*, J. Math. Phys. **26** (1985) 528–531.

[164] J.W. Moffat, *Spinor fields and the $Gl(4,\mathbb{R})$ gauge structure in the nonsymmetric theory of gravitation*, J. Math. Phys. **29** (1988) 1655–1660.

[165] J.W. Moffat, *Review of the nonsymmetric gravitational theory*, Summer Inst. on Gravit., Banff Center, Banff, Alberta, Canada, 1990.

[166] A.M. Naveira, *A classification of Riemannian almost-product manifolds*, Rend. Mat. Appl. (7) **3** (1983) 577–592.

[167] G. 'Olafsson, *Symmetric spaces of Hermitian type*, Differential Geom. Appl. **1** (1991) 195–233.

[168] G. 'Olafsson, *Causal symmetric spaces*, Math. Gött., Heft 15, 1990.

[169] G. 'Olafsson and B. Ørsted, *The holomorphic discrete series for affine symmetric spaces*, J. Funct. Anal. **81** (1988) 126–159.

[170] G.I. Ol'shanskii, *Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series*, Functional Anal. Appl. **15** (1982) 275–285.

[171] G.I. Ol'shanskii, *Convex cones in symmetric Lie algebras, Lie semigroups and invariant causal (order) structures on pseudo-Riemannian symmetric spaces*, Soviet Math. Dokl. **26** (1982) 97–101.

[172] Z. Olszak, *On conformally flat parakählerian manifolds*, Math. Balkanika (N.S.) **5** (1991) 302–307.

[173] Z. Olszak, *Four-dimensional parahermitian manifolds* (preprint).

[174] V. Oproiu, *Some remarkable structures and connections defined on the tangent bundle*, Rend. Mat. Appl. (7) **6** (1973) 503–540.

[175] V. Oproiu, *A pseudo-Riemannian structure in Lagrange geometry*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. a Mat. (N. S.) **33** (1987) 239–254.

[176] V. Oproiu and N. Papaghiuc, *Some classes of almost parahermitian structures on cotangent bundles* (preprint).

[177] T. Oshima and T. Matsuki, *A description of discrete series for semisimple symmetric spaces: Group representations and systems of Differential Equations*, Adv. Stud. in Pure Math. **4** (1984) 331–390, Kinokuniya, Tokyo and North-Holland, Amsterdam.

[178] R.B. Pal and R.S. Mishra, *Hypersurfaces of almost hyperbolic Hermite manifolds*, Indian J. Pure Appl. Math. **11** (1980) 628–632.

[179] D. Papuc, *Sur les variétés des espaces kleinéens à groupe linéaire complétement reductible*, C. R. Acad. Sci. Paris Sér. I Mat. **256** (1963) (1) 62–64; (3) 589–591.

[180] E.M. Patterson, *Riemann extensions which have Kähler metrics*, Proc. Roy. Soc. Edinburgh (A) **64** (1954) 113–126.

[181] E.M. Patterson, *Symmetric Kähler spaces*, J. London Math. Soc. **30** (1955) 286–291.

[182] E.M. Patterson, *A characterization of Kähler manifolds in terms of parallel fields of planes*, J. London Math. Soc. **28** (1958) 260–269.

[183] E. Pavlov, *K-tensors on an almost hyperbolic Hermitian manifold*, Univ. de Plovdiv, Travaux Sci. **22** (1984) Math., 237–247 (Bulgarian, English summary).

[184] D.B. Persic, *Degenerate octaves and projective geometry*, Moskov. Gos. Ped. Inst. Ucen. Zap. **271** (1967) 299–328 (Russian).

[185] D.B. Persic, *Geometries over degenerate octaves*, Dokl. Akad. Nauk. SSSR **173** (1967) 1010–1013 (Russian).

[186] D.B. Persic, *Geometries over the algebra of antioctaves*, Izv. Akad. Nauk SSSR **31** (1967) 1263–1270 (Russian).

[187] D.B. Persic, *On degenerate octaves and degenerate anti-octaves*, Trudy Sem. Vekt. Tenz. Anal. **15** (1970) 165–187 (Russian).

[188] M. Prvanović, *Holomorphically projective transformations in a locally product space*, Math. Balkanika (N.S.) **1** (1971) 195–213.

[189] M. Prvanović, *A note on holomorphically projective transformations of the Kähler spaces*, Tensor (N.S.) **35** (1981) 99–104.

[190] N. Pusic, *On an invariant tensor of a conformal transformation of a hyperbolic Kaehlerian manifold*, Zb. Rad. Filoz. Fak. Nisu, Ser. Mat. **4** (1990) 55–64.

[191] R.P. Rai, *Submanifolds of co-dimension* 2 *of an almost hyperbolic Hermite manifold*, J. Indian Acad. Math. **9** (1987) 1–10.

[192] R.P. Rai and S.D. Singh, *On H-curvature tensors in a hyperbolic Kähler manifold*, Acta Ciencia Indica, Mathematics, **10** (1984) 175–180.

[193] R.P. Rai and S.D. Singh, *H-concircular curvature tensor in a hyperbolic Kähler manifold*, Tamkang J. Math. **16** (1985) 57–70.

[194] P.K. Rashevskij, *The scalar field in a stratified space*, Trudy Sem. Vekt. Tenz. Anal. **6** (1948) 225–248.

[195] P.K. Rashevskij, *Couples de connexions sur une surface à n-dimensions dans une espace stratifiée a 2n-dimensions*, Trudy Sem. Vekt. Tenz. Anal. **8** (1950) 301–313.

[196] E. Reyes, A. Montesinos Amilibia and P.M. Gadea, *Connections making parallel a metric* ($J^4 = 1$)-*structure*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. a Mat. (N. S.) **28** (1982) 49–54.

[197] E. Reyes, A. Montesinos Amilibia and P.M. Gadea, *Connections partially adapted to a metric* ($J^4 = 1$)-*structure*, Colloq. Math. **54** (1987) 216–229.

[198] E. Reyes, A. Montesinos Amilibia and P.M. Gadea, *Projective torsion and curvature, axiom of planes and free mobility for almost product manifolds*, Ann. Polon. Math. **48** (1988) 307–330.

[199] R. Rosca, *Variétés pseudo-riemanniennes* $V^{n,n}$ *de signature (n,n) et á connexion self-orthogonale involutive*, C. R. Acad. Sci. Paris Sér. I Mat. **280** (1974) 959–961.

[200] R. Rosca, *On co-isotropic submanifolds of a para-Kählerian manifold*, Tensor (N.S.) **29** (1975) 200–204.

[201] R. Rosca, *Quantic manifolds with para-co-Kählerian structures*, Kōdai Math. Sem. Rep. **27** (1976) 51–61.

[202] R. Rosca, *Para-Kählerian manifolds carrying a pair of concurrent self-orthogonal vector fields*, Abh. Math. Sem. Univ. Hamburg **46** (1977) 205–215.

[203] R. Rosca, *Espace-temps possédant la propriété géodésique*, C. R. Acad. Sci. Paris Sér. I Mat. **285** (1977) 305–308.

[204] R. Rosca, *Sous-variétés anti-invariantes d'une variété parakählerienne structurée par une connexion géodesique*, C. R. Acad. Sci. Paris Sér. I Mat. **287** (1978) 539–541.

[205] R. Rosca, *Variétés para-cokähleriennes à champ canonique bi-quasi-parallèle*, C. R. Acad. Sci. Paris Sér. I Mat. **289** (1979) 695–698.

[206] R. Rosca, *Sous-variétés pseudo-minimales d'une variété pseudo-riemannienne structurée par une connexion spin-euclidienne*, C. R. Acad. Sci. Paris Sér. I Mat. **290** (1980) 331–333.

[207] R. Rosca, *Para-Kählerian manifolds having the self-orthogonal Killing property*, Riv. Mat. Univ. Parma (4) **7** (1981) 367–376.

[208] R. Rosca, *CR-hypersurfaces à champ normal covariant décomposable incluses dans une variété pseudo-riemannienne neutre*, C. R. Acad. Sci. Paris Sér. I Mat. **292** (1981) 287–290.

[209] R. Rosca, *Variétés pseudo-riemanniennes neutres portant un plan hyperbolique covariant décomposable*, Att. Accad. Sci. Torino **115** (1981) 277–284.

[210] R. Rosca, *Codimension 2, CR-submanifolds with null covariant decomposable vertical distribution of a neutral manifold M*, Rend. Mat. Appl. (7) **2** (1983) 787–797.

[211] R. Rosca, *Pseudo-Euclidean space with Minkowski index and having the quasi-concurrence property*, Tensor (N.S.) **40** (1983) 21–26.

[212] R. Rosca, *CR-sous-variétés co-isotropes d'une variété parakählerienne*, C. R. Acad. Sci. Paris Sér. I Mat. **298** (1984) 149–151.

[213] R. Rosca, *Conformal cosymplectic manifolds endowed with a pseudo-Sasakian structure*, Libertas Math. **4** (1984) 81–94.

[214] R. Rosca, *On pseudo-Sasakian manifolds*, Rend. Mat. Appl. (7) **4** (1984) 394–407.

[215] R. Rosca, *Variétés neutres M admettant une structure conforme symplectique et feuilletage coisotrope*, C. R. Acad. Sci. Paris Sér. I Mat. **300** (1985) 631–634.

[216] R. Rosca, *Exterior concurrent vector fields on a conformal cosymplectic manifold endowed with a Sasakian structure*, Libertas Math. **6** (1986) 167–174.

[217] R. Rosca and L. Vanhecke, *Sur une variété paracokählérienne munie d'une connexion self-orthogonale involutive*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. a Mat. (N. S.) **22** (1976) 49–58.

[218] R. Rosca and L. Vanhecke, *Les sous-variétés isotropes et pseudo-isotropes d'une variété hyperbolique à n dimensions*, Verh. Konink. Acad. Wetensch. Lett. Schone Kunst. Belg. Kl. Wetensch., bf 38 (1976) n. 136.

[219] W. Rossmann, *The structure of semisimple symmetric spaces*, Canad. J. Math. **31** (1979) 157–180.

[220] B.A. Rozenfeld, *History of non-Euclidean geometries*, Springer-Verlag, 1988.

[221] B.A. Rozenfeld, *On unitary and stratified spaces*, Trudy Sem. Vekt. Tenz. Anal. **7** (1949) 260–275 (Russian).

[222] B.A. Rozenfeld, *Projective geometry as metric geometry*, Trudy Sem. Vekt. Tenz. Anal. **8** (1950) 328–354 (Russian).

[223] B.A. Rozenfeld, *Non-Euclidean geometries over the complex and the hypercomplex numbers and their application to real geometries*, in "125 years of Lobatchevski non-Euclidean geometry", Moskow-Leningrad, 1952, 151–156.

[224] B.A. Rozenfeld, *Non-Euclidean geometries*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moskow, 1955, 744 pp. (Russian).

[225] B.A. Rozenfeld and N.N. Adamushko, *The principle of triality in quasi-elliptic and quasi-hyperbolic spaces*, Izv. Vyss. Ucebn. Zaved. Matem. **81** (1969) 79–87 (Russian).

[226] B.A. Rozenfeld, T.A. Burtseva, N.V. Dushina, L.P. Kostrikina and T.I. Yukthina, *Curvature tensors of Hermitian elliptic spaces*, Trudy Geom. Sem. Kazan. Univ. **20** (1990) 85–101 (Russian).

[227] B.A. Rozenfeld, N.V. Dushina and I.N. Semenova, *Differential geometry of real 2-surfaces in dual Hermitian Euclidean and elliptic spaces*, Trudy Geom. Sem. Kazan. Univ. **19** (1989), 107–120 (Russian).

[228] B.A. Rozenfeld, M.P. Zamahahovskii, T.G. Orlovskaja, I.N. Semenova, *Algebras quasisimples, quasimatrices et representations spinorielles des mouvements quasi-euclidiens*, Izv. Vyss. Ucebn. Zaved. Matem. **83** (1969) 62–73 (Russian).

[229] H.S. Ruse, *On parallel fields of planes in a Riemannian manifold*, Quart. J. Math. Oxford **20** (1949) 218–234.

[230] H.S. Ruse, A.G. Walker and T.J. Willmore, *Harmonic spaces*, Cremonese, Roma, 1961.

[231] N. Salingaros, *Algebras with three anticommuting elements. (II) Two algebras over a singular field*, J. Math. Phys. **22** (1981) 2096–2100.

[232] G. Santos-García and P.M. Gadea, *A Weitzenböck formula for the Laplacian of para-Kähler space forms*, Rend. Sem. Math. Messina **2** (1993) 81–89.

[233] G.J. Schellekens, *On a hexagonic structure, I and II*, Indag. Math. **4** (1962) 201–217 and 218–234.

[234] A.P. Shirokov, *On the problem of A-spaces*, in "125 years of Lobatchevski non-Euclidean Geometry", Moskow-Leningrad, 1952, 195–200 (Russian).

[235] A.P. Shirokov, *Geometry of generalized biaxial spaces*, Uc. Zap. K.G.U. **114** (1954) 123–126.

[236] C. Simionescu and O. Banu, *Structures hermitiennes hyperboliques associées a un groupe de transformations presque produit sur un groupe de Lie G_4 resoluble*, Bull. Univ. Brasov, Ser. C, **28** (1986) 97–100.

[237] C. Simionescu and O. Banu, *Structures hermitiennes hyperboliques sur des groupes G_4 resolubles*, XVIIIth Nat. Conf. Geom. Top., Oradea, 1987, Univ. "Babes-Bolyai", Cluj-Napoca, 171–174.

[238] S.S. Singh and P.R. Singh, *On a hyperbolic Kaehler manifold*, J. Nat. Acad. Math. India **5** (1987) 1–9.

[239] B.B. Sinha and Km. Kalpana, *On H-projective curvature tensor in a hyperbolic Kähler manifold*, Progr. Math., Allahabad, **13** (1979) 65–72.

[240] B.B. Sinha and R. Sharma, *Hypersurfaces in an almost paracontact manifold*, Math. Vesnik **4** (1980) 105–112.

[241] E. Study, *The geometry of dynames*, Leipzig, 1903.

[242] M. Takeuchi, *Cell decompositions and Morse equalities on certain symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **12** (1965) 81–191.

[243] M. Takeuchi, *Stability of certain minimal submanifolds of compact Hermitian symmetric spaces*, Tôhoku Math. J. **36** (1984) 293–314.

[244] Y. Tashiro, *On a holomorphically projective correspondence in an almost complex space*, Math. J. Okayama Univ. **6** (1957) 147–152.

[245] M.D. Upadhyay and A.K. Agarwal, *On affine connection in an almost hyperbolic Hermite manifold*, Progr. Math., Allahabad, **15** (1981) 1–8.

[246] M.D. Upadhyay and C.P. Awasthi, *On almost hyperbolic spaces*, Publ. Inst. Math. (Beograd) (N.S.) **20** (1970) 41–49.

[247] M.D. Upadhyay and C.B. Singh, *On almost hyperbolic Hermite manifold*, Demonstratio Math. **15** (1982) 241–247.

[248] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. **24** (1976) 338–351.

[249] I. Vaisman, *Basics of Lagrangian foliations*, Publ. Mat. **33** (1989) 559–575.

[250] R. Vázquez Lorenzo, *Curvatura de variedades para-Kählerianas*, Publ. Dep. Geom. Top. Santiago de Compostela **82**, 1994.

[251] V.V. Vyshnevskij, A.P. Shirokov and V.V. Shurygin, *Spaces over algebras*, Kazan Univ., 1985 (Russian).

[252] A. Weil, *Variétés Kählériennes*, Hermann, Paris, 1958.

[253] C.M. Will, *Violation of the weak equivalence principle in theories of gravity with a nonsymmetric metric*, Phys. Rev. Lett. **62** (1989) 369–372.

[254] J.A. Wolf, *Spaces of constant curvature*, Publish or Perish, 1977.

[255] Y.C. Wong, *Fields of parallel planes in affinely connected spaces*, Quart. J. Math. Oxford **4** (1953) 241–253.

[256] J.C. Wong, *Differential geometry of Grassmann manifolds*, Proc. N.A.S., U.S.A. **57** (1967) 589–594.

[257] G. Vrănceanu and R. Rosca, *Introduction in relativity and pseudo-Riemannian geometry*, Edit. Acad. R.S. Romania, Bucharest, 1976.

[258] I.M. Yaglom, *Projective metric in the plane and complex numbers*, Trudy Sem. Vekt. Tenz. Anal. **7** (1949) 276–318.

[259] I.M. Yaglom, *A simple non-Euclidean geometry and its physical basis*, Springer-Verlag, 1979.

[260] K. Yano and S. Ishihara, *Tangent and cotangent bundles*, Marcel Dekker, 1973.

[261] K. Yano and I. Mogi, *Sur les variétés pseudo-Kaehleriennes à courbure holomorphique constante*, C. R. Acad. Sci. Paris Sér. I Mat. **239** (1953) 962–964.

[262] Z.-Z. Zhong, *On the hyperbolic complex linear symmetry groups and their local gauge transformation actions*, J. Math. Phys. **26** (1985) 404–406.