29 On h-invariant vector fields in a vector bundle

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1. Introduction

In a previous paper [1], we have determined the structure of certain geometrical objects on the total space of the tangent bundle, invariant by the homothety group of the bundle. The relation of these objects with the complete and horizontal lifts for certain objects on the base space were established. The aim of this work is to extend some from those considerations to a general vector bundle. We deal with the vector fields on the total space of a vector bundle which are invariant by the homothety group of the bundle. The set of these fields is a Lie algebra, isomorphic with the Lie algebra of the derivations in the tensor algebra of the considered vector bundle. In particular, if on the vector bundle ξ there exists a flat linear connection, then the Lie algebra of h-invariant vector fields is isomorphic with a certain semidirect product of the Lie algebra of vector fields on the base space by that of tensor fields of type (1,1) on the vector bundle ξ.

2. Definitions and notations

Let be $\xi = (E, \pi, M)$ a C^∞-vector bundle [4], with total space E, projection π and base space M, connected and paracompact. Let be in adapted charts on M, ξ and E, the local coordinates $(x^i), (y^a), (x^i, y^a)$ respectively and the corresponding bases $(\partial_i), (e_a), (\partial_i, \partial_a)$, where $\partial_i = \partial/\partial x^i$, $\partial_a = \partial/\partial y^a$, $i, j, k = 1, 2, ..., m$, $a, b, c = 1, 2, ..., n$. Setting for each $z = (x, y) \in E$, $V_z E = \mathrm{Ker} T_z \pi$, we obtain the vertical distribution and so, the vertical subbundle of TE denoted by VE. Let $\mathcal{F}(M)$ be the ring of C^∞-real functions on M and $f^v = f \circ \pi$, for any $f \in \mathcal{F}(M)$. The set $\mathcal{F}(M)^v = \{f^v \mid f \in \mathcal{F}(M)\}$ is a subring of $\mathcal{F}(E)$, isomorphic with $\mathcal{F}(M)$. Let then $\mathcal{T}^p_q(M)$ and $\mathcal{T}(M)$ be the $\mathcal{F}(M)$-module of tensor fields of type (p, q) and the tensor algebra of M. We denote by $\mathcal{T}^p_q(\xi)$ and $\mathcal{T}(\xi)$ the $\mathcal{F}(M)$-module of tensor fields of type (p, q) and the $\mathcal{F}(M)$-tensor algebra of the bundle ξ.

Definition 2.1. A vector field A on the total space E of the vector bundle ξ is *projectable* [5] if there exists a vector field X on the base space M so that $T\pi(A) = X$.

Such a vector field has the local expression $A = A^i(x)\partial_i + A^a(x, y)\partial_a$. It is easy to see that a projectable vector field may be characterized by the following property:

Proposition 2.1. *A vector field A on the total space E of the vector bundle ξ is projectable if and only if the Lie derivative $\mathcal{L}_A$ preserves the subbundle $V\mathcal{T}(\xi)$ of vertical vector fields on E.*

The set $\mathcal{P}^1(E)$ of projectable vector fields on E is a $\mathcal{F}(M)$-Lie algebra, which is a $I\!R$-subalgebra of $\mathcal{T}^1(E)$. Setting for each 1-form $\mu \in \mathcal{T}_1(\xi)$, given locally by $\mu(x) = \mu_a(x)e^a$,

$$\gamma(\mu)(z) = \mu_a(x)y^a, \tag{1}$$

where $z = (x, y) \in E$, we obtain a class of functions on E with the property that every vector field $A \in \mathcal{T}^1(E)$ is uniquely determined by its values on the functions of this type.

Definition 2.2. The *canonical* vector field on the total space E is the vector field K given by

$$K(\gamma\mu) = \gamma\mu, \ \forall \mu \in \mathcal{T}_1(\xi). \tag{2}$$

It has the local expression $K(z) = y^a\partial_a$.

The mapping γ may be extended to tensor fields $S \in \mathcal{T}_1^1(\xi)$ by

$$\gamma S(\gamma\mu) = \gamma(\mu \circ S), \ \forall \mu \in \mathcal{T}_1(\xi). \tag{3}$$

If S has the local expression $S(x) = S_b^a(x)e_a \otimes e^b$, then $\gamma S(z) = S_b^a(x)y^b\partial_a$ that is, γS is a vertical vector field on E. Setting $S = I$, the identical automorphism of ξ, we obtain a new characterization for the canonical vector field

$$K = \gamma(I). \tag{4}$$

Definition 2.3. The *vertical lift* of a section $u \in \mathcal{T}^1(\xi)$ is the vertical vector field on E given by

$$u^v(\gamma\mu) = \mu(u)^v, \ \forall \mu \in \mathcal{T}_1(\xi). \tag{5}$$

If u has the local expression $u(x) = u^a(x)e_a$, then $u^v(z) = u^a(x)\partial_a$.

Remark 2.1. The vertical lift may be considered as a natural isomorphism $v:\pi^*E{=}E{\times}_M E{\to}VE$ given by $v(z, u) = u^v(z)$.

For the vertical lift and the mapping γ we notice the properties:

$$(fu + gv)^v = f^v u^v + g^v v^v, [u^v, v^v] = 0, \mathcal{L}_K u^v = -u^v, [\gamma S, \gamma T] = \gamma[T, S] \tag{6}$$

where $f, g \in \mathcal{F}(M), \ u, v \in \mathcal{T}^1(\xi), \ S, T \in \mathcal{T}_1^1(\xi)$.

3. h-Invariant vector fields on E and derivations in ξ

Definition 3.1. The *homothety group* of the vector bundle ξ is the 1-parameter group of transformations on E given by

$$h_t(x, y) = (x, ty), \ \forall (x, y) \in E, \ t \in I\!R^*. \tag{7}$$

Definition 3.2. A *h-invariant* vector field is a vector field on E which is invariant by the homotheties of ξ.

It is easy to see that the vector field generated by the homothety group of ξ is the canonical vector field K and so it follows [2].

Proposition 3.1. *A vector field $A \in \mathcal{T}^1(E)$ is h-invariant if and only if*

$$\mathcal{L}_K A = 0. \tag{8}$$

Setting locally, $A(z) = A^i(x,y)\partial_i + A^a(x,y)\partial_a$, we get

$$\mathcal{L}_K A = \frac{\partial A^i}{\partial y^b} y^b \partial_i + \left(\frac{\partial A^a}{\partial y^b} y^b - A^a \right) \partial_a \tag{9}$$

and therefore we obtain:

Proposition 3.2. *A vector field $A \in \mathcal{T}^1(E)$ is h-invariant if and only if its local coordinates are given by*

$$A^i = A^i(x), \ A^a = A^a_b(x) y^b. \tag{10}$$

Remark 3.2. Every h-invariant vector field is projectable.

Taking into account the law of transformation for the coordinates of a vector field on E under the change of coordinates, given by

$$x^{i'} = x^{i'}(x^i), \ y^{a'} = M^{a'}_a(x) y^a, \tag{11}$$

we obtain for A^i and A^a_b

$$A^{i'} = A^i \frac{\partial x^{i'}}{\partial x^i}, \ A^{a'}_{b'} = A^a_b M^{a'}_a M^b_{b'} + A^i \frac{\partial M^{a'}_b}{\partial x^i} M^b_{b'}. \tag{12}$$

On the other hand, considering a derivation D in the tensor algebra $\mathcal{T}(\xi)$, which preserves the type and commutes with the contractions and setting in the corresponding chart on ξ,

$$D(x^i) = D^i, \quad D(e_b) = D^a_b e_a, \tag{13}$$

we obtain to the change of the chart

$$D^{i'} = D^i \frac{\partial x^{i'}}{\partial x^i}, \quad D^{a'}_{b'} = D^a_b M^{a'}_a M^b_{b'} - D^i \frac{\partial M^{a'}_b}{\partial x^i} M^b_{b'}. \tag{14}$$

From (12) and (14), it follows that setting

$$A^i = D^i, \ A^a_b = -D^a_b, \tag{15}$$

we obtain a bijection between the set $\mathcal{D}er\mathcal{T}(\xi)$ of the derivations in the tensor algebra $\mathcal{T}(\xi)$ and the set $\mathcal{H}^1(E)$ of the h-invariant vector fields on E. By this bijection, to a derivation $D = iS$ defined by a tensor field $S \in \mathcal{T}^1_1(\xi)$ it corresponds the vertical h-invariant vector field $A = -\gamma S \in V\mathcal{H}^1(E)$. From (15) it results that the bijection between $\mathcal{D}er\,\mathcal{T}(\xi)$ and $\mathcal{H}^1(E)$, which associates to the derivation D the h-invariant vector field A, is given, in an invariant form, by

$$A(\gamma\mu) = \gamma(D\mu), \quad \forall \mu \in \mathcal{T}_1(\xi). \tag{16}$$

Thus, we can give the following

Definition 3.3. The *complet lift* of a derivation $D \in \mathcal{D}er\,\mathcal{T}(\xi)$ is the h-invariant vector field $D^c = A \in \mathcal{H}^1(E)$ given by the formula (16).

From (16) we obtain for the complet lift $c : \mathcal{D}er\,\mathcal{T}(\xi) \longrightarrow \mathcal{H}^1(E)$ the following properties:

$$\begin{array}{l} D^c(f^v) = (Df)^v, \ (f_1 D_1 + f_2 D_2)^c = f_1^v D_1^c + f_2^v D_2^c, \\ {[D_1, D_2]}^c = [D_1^c, D_2^c], \ (iS)^c = -\gamma S, \ [iS, iT]^c = \gamma[T,S], \\ {[D^c, K]} = 0, \quad [D^c, u^v] = (Du)^v, \quad [D^c, \gamma S] = \gamma(DS). \end{array} \tag{17}$$

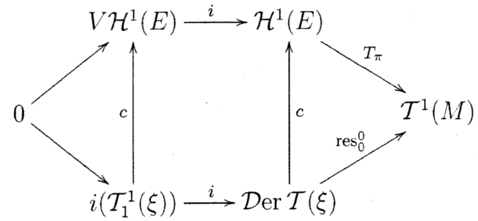
Summarizing, we have

Theorem 3.1. (i) *$\mathcal{H}^1(E)$ is a Lie subalgebra of $\mathcal{P}^1(E)$.*

(ii) *The complete lift* $c : \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi) \longrightarrow \mathcal{H}^1(E)$ *is an isomorphism of* $I\!R$*-Lie algebras, which carries the ideal* $i(\mathcal{T}_1^1(\xi))$ *in* $V\mathcal{H}^1(E)$.

Remark 3.3. Denoting by res_q^p the map which associates to each derivation $D \in \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$ its restriction to $\mathcal{T}_q^p(\xi)$, we obtain the following commutative diagram

(18)



4. Linear connections and derivations on ξ

Let be ∇ a linear connection on the vector bundle ξ considered as a $\mathcal{F}(M)$–linear mapping $\nabla : \mathcal{T}^1(M) \longrightarrow \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$, which satisfies the property $(\mathrm{res}_0^0 \circ \nabla)(X) = X$, for each $X \in \mathcal{T}^1(M)$.

Definition 4.1. The *horizontal lift* of a vector field $X \in \mathcal{T}^1(M)$, with respect to linear connection ∇ on ξ, is the h-invariant vector field X^h on E, given by

(19) $$X^h = (\nabla_X)^c.$$

From (16) we obtain

Proposition 4.1. *The horizontal lift of the vector field* $X in \mathcal{T}^1(M)$, *with respect to linear connection* ∇ *on* ξ, *is the vector field* $X^h \in \mathcal{H}^1(E)$ *given by*

(20) $$X^h(\gamma\mu) = \gamma(\nabla_X \mu), \quad \forall \mu \in \mathcal{T}_1(\xi).$$

We notice the following properties of the horizontal lift:

(21) $$\begin{array}{l} X^h(f^v) = (Xf)^v, (fX+gY)^h = f^v X^h + g^v Y^h, [X^h, Y^h] = [X,Y]^h - \gamma R_{XY} \\ [X^h, u^v] = (\nabla_X u)^v, [X^h, K] = 0, [X^h, \gamma S] = \gamma \nabla_X S, \end{array}$$

where R is the curvature of ∇.

Considering a linear connection ∇ on ξ as a splitting for the following exact sequence of $\mathcal{F}(M)$-modules

(22) $$0 \longrightarrow i(\mathcal{T}_1^1(\xi)) \overset{i}{\longrightarrow} \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi) \underset{\nabla}{\overset{\mathrm{res}_0^0}{\rightleftarrows}} \mathcal{T}^1(M) \longrightarrow 0,$$

we obtain that each $D \in \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$ may be uniquely decomposed in the form

(23) $$D = \nabla_X + i(S),$$

where $X = \mathrm{res}_0^0(D)$ and $S = \mathrm{res}_0^1(D - \nabla_X)$. It follows

Proposition 4.2. *Given a linear connection ∇ on the vector bundle ξ, the map $\nabla \times i : \mathcal{T}^1(M) \times \mathcal{T}_1^1(\xi) \longrightarrow \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$ given by*

$$\nabla \times i\,(X, S) = \nabla_X + i(S), \tag{24}$$

is an isomorphism of $\mathcal{F}(M)$-modules.

Considering two derivations $D_\alpha = \nabla_{X_\alpha} + i(S_\alpha)$, $\alpha = 1, 2$, we obtain from (23)

$$[D_1, D_2] = \nabla_{[X_1, X_2]} + i([S_1, S_2] + \nabla_{X_1} S_2 - \nabla_{X_2} S_1 + R_{X_1 X_2}). \tag{25}$$

Therefore, the map $\nabla \times i$ is not an isomorphism between the direct product of Lie algebras $\mathcal{T}^1(M) \times \mathcal{T}_1^1(\xi)$ and $\mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$. Let us suppose that there exists a semidirect product of Lie algebras $\mathcal{T}^1(M)$ and $\mathcal{T}_1^1(\xi)$, given by a morphism $\rho : \mathcal{T}^1(M) \longrightarrow \mathrm{Der}\,\mathcal{T}_1^1(\xi)$ so that $\nabla \times i : \mathcal{T}^1(M) \times_\rho \mathcal{T}_1^1(\xi) \longrightarrow \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$ to becomes an isomorphism. From the definition of the semidirect product, we must have

$$[(X_1, S_1), (X_2, S_2)] = ([X_1, X_2], [S_1, S_2] + \rho_{X_1} S_2 - \rho_{X_2} S_1). \tag{26}$$

Then $\nabla \times i$ being an isomorphism, one has

$$(\nabla \times i)([(X_1, S_1), (X_2, S_2)]) = [D_1, D_2]. \tag{27}$$

Thus, taking into account (26),(24) and (25), we obtain

$$\rho_{X_1} S_2 - \rho_{X_2} S_1 = \nabla_{X_1} S_2 - \nabla_{X_2} S_1 + R_{X_1 X_2}, \ \forall X_1, X_2 \in \mathcal{T}^1(M), S_1, S_2 \in \mathcal{T}_1^1(\xi).$$

Setting here $S_1 = S_2 = 0$, it follows $R = 0$. Next, putting $S_1 = 0$, we get $\rho_{X_1} S_2 = \nabla_{X_1} S_2$, $\forall X_1 \in \mathcal{T}^1(M)$, $S_2 \in \mathcal{T}_1^1(\xi)$, that is

$$\rho = \mathrm{res}_1^1 \circ \nabla. \tag{28}$$

From $R = 0$, it results that ρ, given by this formula, is a morphism of Lie algebras. We have obtained

Proposition 4.3. *Given a linear connection ∇ on the vector bundle ξ, the map $\nabla \times i : \mathcal{T}^1(M) \times_\rho \mathcal{T}_1^1(\xi) \longrightarrow \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$, defined by (24) is an isomorphism of Lie algebras if and only if ∇ is flat and $\rho = \mathrm{res}_1^1 \circ \nabla$.*

Considering now a derivation $D \in \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$, given by (24) and taking its complete lift, we obtain

$$D^c = X^h - \gamma S, \ X \in \mathcal{T}^1(M), \ S \in \mathcal{T}_1^1(\xi). \tag{29}$$

It follows from here

Proposition 4.4. *Given a linear connection ∇ on the vector bundle ξ, every h-invariant vector field $A \in \mathcal{H}^1(E)$ can be uniquely decomposed in the form (2.9) and the map $h \times (-\gamma) : \mathcal{T}^1(M) \times \mathcal{T}_1^1(\xi) \longrightarrow \mathcal{H}^1(E)$, given by*

$$(h \times (-\gamma))(X, S) = X^h - \gamma S, \tag{30}$$

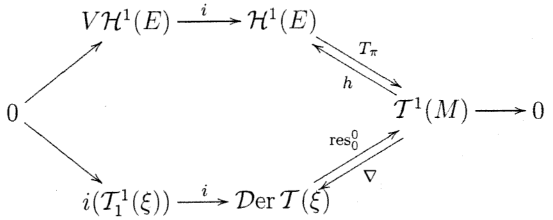
is an isomorphism of $\mathcal{F}(M)$-modules.

Finally, taking into account the two last propositions, we get

Theorem 4.1. *Given a linear connection* ∇ *on the vector bundle* ξ*, the map* $h \times (-\gamma) = c \circ (\nabla \times i) : \mathcal{T}^1(M) \times_\rho \mathcal{T}_1^1(\xi) \longrightarrow \mathcal{H}^1(E)$ *is an isomorphism of Lie algebra if and only if* ∇ *is flat and* $\rho = \mathrm{res}_1^1 \circ \nabla$.

Remark 4.1. Considering a linear connection ∇ on ξ, the sequences

(31)



are generally splitting exact sequences of $\mathcal{F}(M)$-modules with the right splittings $h : \mathcal{T}^1(M) \longrightarrow \mathcal{H}^1(E)$ and $\nabla : \mathcal{T}^1(M) \longrightarrow \mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$. They are splitting exact sequences of $I\!R$-Lie algebras if and only if the connection ∇ is flat.

Remark 4.2. From the diagram (31) it follows $c \circ \nabla = h$. Because c is a bijection one has $\nabla = c^{-1} \circ h$. Thus, we obtain a new definition for a linear connection on a vector bundle.

Definition 4.2. A linear connection on the vector bundle $\xi = (E, \pi, M)$ is a right splitting h for the short sequence of $\mathcal{F}(M)$-modules

$$0 \longrightarrow V\mathcal{H}^1(E) \xrightarrow{\ i\ } \mathcal{H}^1(E) \underset{h}{\overset{T\pi}{\rightleftarrows}} \mathcal{T}^1(M) \longrightarrow 0.$$

For the associated covariant derivation we obtain from (17)

$$\nabla_X u = v^{-1}[h(X), u^v], \ \forall X \in \mathcal{T}^1(M), \ u \in \mathcal{T}^1(\xi).$$

Remark 4.3. Associating to each h-invariant vector field $A = D^c$, the Lie derivative $\mathcal{L}_A$, we may consider the complete lift c as a mapping from $\mathcal{D}\mathrm{er}\,\mathcal{T}(\xi)$ to $\mathcal{D}er\,\mathcal{T}(E)$ or to $\mathcal{D}er\,V\mathcal{T}(E)$.

Remark 4.4. If ξ is the tangent bundle of M, then $\mathcal{T}^1(\xi) = \mathcal{T}^1(M)$ is a Lie algebra and so we may associate in a canonical manner, to a vector field $X \in \mathcal{T}^1(M)$, the Lie derivative $\mathcal{L}_X$. Then, the complete lift $(L_X)^c$ is a h-invariant vector field on $E = TM$, called the *complete lift* of X and denoted by X^c. Therefore, the possibility for defining the complete lift X^c for a vector field X on the tangent bundle is the consequence of the fact that $\mathcal{T}^1(M)$ is a Lie algebra.

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