30 On almost para-Hermitian manifolds

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1. Introduction

In the last time the almost para-Hermitian (aph) structures are researched by many geometers for their interesting properties and useful applications in the theoretical physics (see for example [4] and the references therein). In the study of these structures is used generally the Levi-Civita connection of the associated metric. But, as in the Hermitian case, it is possible to consider a canonical connection, strongly related to the aph- structure and which was introduced by P. Libermann [8].

In this work, we give an invariant study for the canonical connection and we use it to obtain some simple characterisations for certain classes of aph- structures. The organisation of the paper is as follows: in the second section we present the basic result about almost para-Hermitian (aph) manifolds; in the third section we study the canonical connection, proving that it is invariant under the action of a certain group of automorphisms of the module of connections; in the fourth one we obtain some important classes of aph-structures and the last three sections are devoted to other connections compatible with an aph-structure, the expressions in local coordinates and the examples.

2. Basic results

We remember some definitions and results concerning paracomplex manifolds, compatible connections and para-Hermitian structures.

Let M be a connected and paracompact C^∞-manifold, $\mathcal{F}(M)$ the ring of real functions, $\mathcal{T}_q^p(M)$ the $\mathcal{F}(M)$-module of (p,q) tensor fields and $\mathcal{T}(M)$ the $\mathcal{F}(M)$-tensor algebra of M. Denote by TM the total space of the tangent bundle of M.

Definition 2.1. An *almost paracomplex* (apc) structure on the manifold M is a tensor field $F \in \mathcal{T}_1^1(M)$ which satisfies the conditions

$$F^2 = I,\ TrF = 0. \tag{1}$$

An *almost paracomplex* (apc) manifold is a manifold endowed with an apc-structure.

It follows that $dimM = 2n$ and F is a particular almost product structure on M, with the projectors

(2) $$F_1 = \frac{I+F}{2};\ F_2 = \frac{I-F}{2}$$

and the eigendistributions (vector subbundles of TM)

(3) $$V_1 = F_1(TM),\ V_2 = F_2(TM),\ dimV_1 = dimV_2 = n.$$

Definition 2.2. A *paracomplex* (pc) structure on M is an apc- structure F with the property that the distributions V_1 and V_2 are involutive.

Using a result obtained by one of us in a more general setting [2], we obtain

Proposition 2.3. *The set of connections ∇ on M which are compatible with an apc-structure F (i.e., $\nabla F = 0$) is given by*

(4) $$\nabla = \Phi_F(\overset{\circ}{\nabla}) + \Psi_F(\sigma),$$

where $\overset{\circ}{\nabla}$ is a fixed arbitrary connection, σ is any $(1,2)$-tensor field on M and $\forall X \in T^1(M)$

(5) $$\Phi_F(\overset{\circ}{\nabla})_X = \frac{1}{2}(\overset{\circ}{\nabla}_X + F \circ \overset{\circ}{\nabla}_X \circ F), \Psi_F(\sigma)_X = \frac{1}{2}(\sigma_X + F \circ \sigma_X \circ F).$$

As in the complex case [7] one obtains

Proposition 2.4. *An apc-structure F on M is paracomplex if and only if it satisfies one of the conditions:*

a) *The Nijenhuis tensor N of F vanishes (i.e., F is integrable)*

b) *There exists a symmetric F-connection ∇ on M (i.e., $\nabla F = 0$).*

Definition 2.5. An *almost para-Hermitian* (aph) structure on the manifold M is a pair (F, G), where F is an almost product structure and G is a pseudo-Riemannian metric on M related by one of the following equivalent compatibility conditions

(6) $$G \circ (F \times F) = -G,\ G \circ (F \times I) = -G \circ (I \times F).$$

An *almost para-Hermitian* (aph) manifold is a manifold endowed with an aph-structure (F, G). The 2-form on M given by

(7) $$\Omega = G \circ (F \times I)$$

is called the *fundamental* 2-form associated to the aph-structure (F, G).

Ω defines an almost symplectic structure on M, which satisfies the condition

(8) $$\Omega \circ (F \times F) = -\Omega.$$

It follows from (6) $TrF = 0$, i.e. F defines an apc-structure on M, $signG = (n, n)$, i.e. G defines a *neutral* structure (metric) on M and $dimM = 2n$. From (6) and (8) it results that the eigendistributions V_1 and V_2 are maximal isotropic for G and Ω, i.e. $G|_{V_1} = G|_{V_2} = 0$, $\Omega|_{V_1} = \Omega|_{V_2} = 0$.

3. The canonical connection on an aph-manifold

Beside the Levi-Civita connection of G, used by many authors in the study of an aph-structure, there exists another connection, very convenient for such a structure given by

Proposition 3.1. *For an aph-structure (F, G) on a manifold M there exists an unique connection ∇ with torsion T which satisfies the conditions:*

$$\nabla F = 0,\ \nabla G = 0,\ T \circ (F_1 \times F_2) = 0. \tag{9}$$

Proof. *Uniqueness:* Considering V_1 and V_2 as subbundles of TM, from $\nabla F = 0$ one obtains

$$\nabla_X Y_i \in \mathcal{T}^1(M, V_i),\ \forall X \in \mathcal{T}^1(M), Y_i \in \mathcal{T}^1(M, V_i), i = 1, 2, \tag{10}$$

that is, V_1 and V_2 are invariant to parallel transport defined by ∇ on M.

Then, since $T \circ (F_1 \times F_2) = 0$, we get

$$\nabla_{X_1} X_2 = F_2[X_1, X_2],\ \nabla_{X_2} X_1 = F_1[X_2, X_1],\ \forall X_i \in \mathcal{T}^1(M, V_i), i = 1, 2. \tag{11}$$

Finally, taking account of $\nabla G = 0$, we obtain

$$\begin{aligned} G(\nabla_{X_1} Y_1, Z_2) &= X_1 G(Y_1, Z_2) - G([X_1, Z_2], Y_1) \\ G(\nabla_{X_2} Y_2, Z_1) &= X_2 G(Y_2, Z_1) - G([X_2, Z_1], Y_2). \end{aligned} \tag{12}$$

From (11) and (12) it results that ∇ is unique.

Existence: Considering $\nabla : \mathcal{T}^1(M) \times \mathcal{T}^1(M) \to \mathcal{T}^1(M)$ given by (11) and (12) and setting $\nabla_X f = X(f)$ for $f \in \mathcal{F}(M)$, it is easy to check that ∇ defines a connection on M which satisfies (9). □

Definition 3.2. We shall call the *canonical connection* associated to aph-structure (F, G), the connection given by (11) and (12).

Since V_1 and V_2 are invariant to parallel transport defined by ∇ on M, then setting

$$\overset{1}{\nabla}_X Y_1 = \nabla_X Y_1,\ \overset{2}{\nabla}_X Y_2 = \nabla_X Y_2,\quad \forall X \in \mathcal{T}^1(M),\ Y_i \in \mathcal{T}^1(M, V_i),\ i = 1, 2, \tag{13}$$

we obtain linear connections $\overset{i}{\nabla}$ for each bundle V_i. We call the *torsion* of the connection $\overset{i}{\nabla}$, the tensor field

$$\overset{i}{T} = F_i \circ T \circ (F_i \times F_i),\ i = 1, 2, \tag{14}$$

restricted to V_i and we obtain

$$\overset{i}{T}(X_i, Y_i) = \overset{i}{\nabla}_{X_i} Y_i - \overset{i}{\nabla}_{Y_i} X_i - F_i[X_i, Y_i],\ i = 1, 2.$$

Setting then

$$\overset{1}{S} = F_2 \circ T \circ (F_1 \times F_1),\ \overset{2}{S} = F_1 \circ T \circ (F_2 \times F_2), \tag{15}$$

one gets

$$\overset{1}{S}(X_1, Y_1) = -F_2[X_1, Y_1],\ \overset{2}{S}(X_2, Y_2) = -F_1[X_2, Y_2]. \tag{16}$$

We call $\overset{1}{S}$ and $\overset{2}{S}$ the *relative torsions* or *tensors of non-holonomy* for the distributions V_1 and V_2. We remark that the condition $T \circ (F_1 \times F_2) = 0$ is equivalent with $T \circ (F \times F) = T$.

Let be $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ the curvature tensor field of ∇ and $\widetilde{R}_{XYZW} = G(R_{XY}Z, W)$. From $\nabla F = 0$, $\nabla G = 0$ and $\nabla \Omega = 0$ it follows respectively

$$R_{XY} \circ F = F \circ R_{XY},\ \widetilde{R}_{XYZW} = -\widetilde{R}_{XYWZ},\ \widetilde{R}_{XYF(Z)W} = -\widetilde{R}_{XYZF(W)}. \tag{17}$$

Considering $\widetilde{R}_{XY}$ as a $(0,2)$ tensor field on M, we find

$$\widetilde{R}_{XY} \circ (F \times F) = -\widetilde{R}_{XY} \tag{18}$$

and therefore $\widetilde{R}_{XY}\ |_{V_1} = \widetilde{R}_{XY}\ |_{V_2} = 0$. Denoting by D_F the derivative in $\mathcal{T}(M)$ defined by F, we obtain from (6), (8), (17) and (18)

$$D_F G = 0,\ D_F \Omega = 0,\ D_F R_{XY} = 0,\ D_F \widetilde{R}_{XY} = 0. \tag{19}$$

Setting then

$$\tau_t = I \cosh t + F \sinh t,\ t \in \mathbf{R}, \tag{20}$$

we obtain a 1-parameter group of automorphisms for the $\mathcal{F}(M)$-module $\mathcal{T}^1(M)$. It may be extended to tensor algebra $\mathcal{T}(M)$, by putting

$$\begin{gathered} \tau_t(f) = f,\ \tau_t(\omega) = \omega \circ \tau_t^{-1}, \\ \tau_t(T)(\omega^1, \dots, X_1, \dots) = T(\omega^1 \circ \tau_t, \dots, \tau_t^{-1}(X_1), \dots), \end{gathered} \tag{21}$$

where $f \in \mathcal{F}(M)$, $\omega^i \in \mathcal{T}_1(M)$, $X_i \in \mathcal{T}^1(M)$ and $T \in \mathcal{T}_q^p(M)$. From here, we obtain the following geometrical meaning for the relations (19):

Proposition 3.3. *The tensor fields G, Ω, R_{XY} and $\widetilde{R}_{XY}$ are invariant under the action of the group* (20) *on the algebra $\mathcal{T}(M)$, given by* (21).

Putting for a linear connection ∇ on M

$$\tau_t(\nabla)_X = \tau_t \circ \nabla_X \circ \tau_t^{-1},\ t \in \mathbf{R}, \tag{22}$$

we obtain a group of automorphisms for the $\mathcal{F}(M)$-affine module $\mathcal{C}(M)$ of connections on M [2] and from $\nabla F = 0$ it follows

Proposition 3.4. *The canonical connection ∇ of the aph-structure (F, G) is invariant under the action* (22) *of the group* (20) *on the affine module $\mathcal{C}(M)$.*

For the Nijenhuis tensor field N of F given by

$$N(X, Y) = [X, Y] - F[FX, Y] - F[X, FY] + [FX, FY]$$

we obtain

$$N(X_1,Y_1)=4F_2[X_1,Y_1],\ N(X_1,Y_2)=0,\ N(X_2,Y_2)=4F_1[X_2,Y_2] \tag{23}$$

and taking account of (16), it follows

$$N(X_1,Y_1)=-4\overset{1}{S}(X_1,Y_1),\ N(X_1,Y_2)=0,\ N(X_2,Y_2)=-4\overset{2}{S}(X_2,Y_2). \tag{24}$$

Finally, for the exterior derivative of the fundamental 2-form Ω, taking account of $\nabla\Omega=0$, we get

$$3d\Omega(X,Y,Z)=\sum_{XYZ}\Omega(T(X,Y),Z),$$

(where $\sum_{XYZ}$ denotes the cyclic sum) and from here it follows

$$\begin{aligned} &3d\Omega(X_1,Y_1,Z_1)=\sum_{X_1Y_1Z_1}\Omega(\overset{1}{S}(X_1,Y_1),Z_1), \quad 3d\Omega(X_1,Y_1,Z_2)=\Omega(\overset{1}{T}(X_1,Y_1),Z_2),\\ &3d\Omega(X_2,Y_2,Z_1)=\Omega(\overset{2}{T}(X_2,Y_2),Z_1), \quad 3d\Omega(X_2,Y_2,Z_2)=\sum_{X_2Y_2Z_2}\Omega(\overset{2}{S}(X_2,Y_2),Z_2). \end{aligned} \tag{25}$$

4. Some important classes of aph-structures

From the properties of the canonical connection ∇ associated to an aph-structure on the manifold M, established up to here, we shall see that the torsion T of this connection is very important for the characterisation of certain aph-structures. Since V_1 and V_2 have a symmetrical position in an aph structure, we shall enumerate only the classes relative to V_1 or to V_1 and V_2 simultaneously.

Definition 4.1. We shall say that an aph-structure on M is:

a) 1-*para-Hermitian* if and only if the distribution V_1 is involutive,
b) *para-Hermitian* if and only if V_1 and V_2 are involutive,
c) 1-*almost para-Kählerian* if and only if $i_{X_1}i_{Y_1}d\Omega=0$,
d) *almost para-Kählerian* if and only if $d\Omega=0$,
e) 1-*para-Kählerian* if and only if V_1 is involutive and $i_{X_1}i_{Y_1}d\Omega=0$,
f) *para-Kählerian* if and only if it is para-Hermtian and almost para- Kählerian.

From the previous considerations, we obtain

Proposition 4.2. *An aph-structure (F,G) on the manifold M is:*

a) 1-*para-Hermitian if and only if* $N\mid_{V_1}=0$, *or* $\overset{1}{S}=0$, *or* $D_F(i_{X_1}T)=0$, $\forall X_1\in\mathcal{T}^1(M,V_1)$,

b) *para-Hermitian if and only if* $N=0$, *or* $\overset{1}{S}=\overset{2}{S}=0$, *or* $D_F(i_XT)=0$, $\forall X\in\mathcal{T}^1(M)$,

c) 1-*almost para-Kählerian if and only if* $\sum_{X_1Y_1Z_1}\Omega(\overset{1}{S}(X_1,Y_1),Z_1)=0$, $\overset{1}{T}=0$,

d) *almost para-Kählerian if and only if*

$$\textstyle\sum_{X_1Y_1Z_1}\Omega(\overset{1}{S}(X_1,Y_1),Z_1)=0,\ \overset{1}{T}=0,\quad \sum_{X_2Y_2Z_2}\Omega(\overset{2}{S}(X_2,Y_2),Z_2)=0,\ \overset{2}{T}=0,$$

e) 1-*para-Kählerian if and only if* $\overset{1}{S} = \overset{1}{T} = 0$, *or* $T\,|_{V_1} = 0$, *or* $i_{X_1}T = 0$,

f) *para-Kählerian if and only if* $N = 0$, $d\Omega = 0$, *or* $T = 0$, *or* $\nabla = \tilde{\nabla}$, *where* $\tilde{\nabla}$ *is the Levi-Civita connection of* G.

5. Other connections compatible with an aph-structure

From the work [2] it follows

Proposition 5.1. *The set of connections on* M *compatible with the aph-structure* (F, G) *is given by*

$$\nabla = \Phi_F \circ \Phi_G(\overset{\circ}{\nabla}) + \Psi_F \circ \Psi_G(\sigma), \tag{26}$$

where $\overset{\circ}{\nabla}$ *is an arbitrary fixed connection,* σ *any* $(1,2)$*-tensor field on* M, Φ_F *and* Ψ_F *are given by* (5) *and*

$$\begin{aligned} \Phi_G(\overset{\circ}{\nabla})_X &= \frac{1}{2}(\overset{\circ}{\nabla}_X + G^{-1} \circ \overset{\circ}{\nabla}_X \circ G), \\ \Psi_G(\sigma)_X &= \frac{1}{2}(\sigma_X + G^{-1} \circ \sigma_X \circ G), \end{aligned} \qquad \forall X \in \mathcal{T}^1(M). \tag{27}$$

Taking here $\nabla = \tilde{\nabla}$, the Levi-Civita connection of G and setting $\sigma = 0$, we obtain

$$\Phi_G(\tilde{\nabla}) = \tilde{\nabla},\ \Phi_F(\tilde{\nabla})_X = \frac{1}{2}(\tilde{\nabla}_X + F \circ \tilde{\nabla}_X \circ F),\ \Psi_F \circ \Psi_G(0) = 0$$

and so we get

Proposition 5.2. *If* $\tilde{\nabla}$ *is the Levi-Civita connection of* G, *setting*

$$D_X = \frac{1}{2}(\tilde{\nabla}_X + F \circ \tilde{\nabla}_X \circ F), \tag{28}$$

then D *is a connection on* M *compatible with the aph-structure* (F, G).

The connection D will be called the *natural* connection associated to the aph-structure (F, G) on M.

From (28) it follows

$$D_X Y_1 = F_1 \tilde{\nabla}_X Y_1,\ D_X Y_2 = F_2 \tilde{\nabla}_X Y_2,\ \forall X \in \mathcal{T}^1(M),\ Y_i \in \mathcal{T}^1(M, V_i) \tag{29}$$

and so

$$D_X = F_1 \circ \tilde{\nabla}_X \circ F_1 + F_2 \circ \tilde{\nabla}_X \circ F_2. \tag{30}$$

Hence, the restrictions of D to the subbundles V_1 and V_2 coincide with the projections of $\tilde{\nabla}$ to V_1 and V_2. For the torsion τ of D we obtain

$$\begin{aligned} \tau(X_1, Y_1) = \overset{1}{S}(X_1, Y_1),\ \tau(X_2, Y_2) = \overset{2}{S}(X_2, Y_2) \\ \tau(X_1, Y_2) = F_2 \tilde{\nabla}_{Y_2} X_1 - F_1 \tilde{\nabla}_{X_1} Y_2. \end{aligned} \tag{31}$$

thus showing that the natural connection D is more complicate than the canonical connection ∇.

It is easy to see that if ∇ is an arbitrary metric connection for G on M, with torsion T, and $\tilde{\nabla}$ is the Levi-Civita connection of G, then one has

$$G(\nabla_X Y, Z) = G(\tilde{\nabla}_X Y, Z) + \frac{1}{2}\{G(T(X,Y),Z) - G(T(Y,Z),X) + G(T(Z,X),Y)\}.$$

Taking here for ∇ the canonical connection of (F,G), we obtain

$$\nabla_{X_1} Y_1 = F_1 \tilde{\nabla}_{X_1} Y_1 + \frac{1}{2}\overset{1}{T}(X_1, Y_1),\ \nabla_{X_2} Y_2 = F_2 \tilde{\nabla}_{X_2} Y_2 + \frac{1}{2}\overset{2}{T}(X_2, Y_2) \tag{32}$$

and so it follows

Proposition 5.3. *The natural connection D given by* (28) *coincides with the canonical connection ∇ for the aph-structure (F,G) if and only if $\overset{1}{T} = \overset{2}{T} = 0$ or $\tau \circ (F_1 \times F_2) = 0$. The natural connection D coincides with the Levi-Civita connection $\tilde{\nabla}$ of G if and only if $T = 0$ or $\tau = 0$ or $\tilde{\nabla} F = 0$, i.e., in the case of a para-Hermitian structure (F,G) on M.*

6. Expressions in local coordinates

In the case of a para-Hermitian structure (F,G), the distributions V_1 and V_2 being involutive, we can choose the local coordinates $(x^i, x^{\bar{\imath}})$, $i = 1, \dots n$, $\bar{\imath} = n + i$, so that the leaves of V_1 and V_2 will be given respectively by

$$x^{\bar{\imath}} = \text{const.},\ x^i = \text{const.} \tag{33}$$

Setting $e_i = \dfrac{\partial}{\partial x^i}$, $e_{\bar{\imath}} = \dfrac{\partial}{\partial x^{\bar{\imath}}}$, from (3) and (6) we obtain for F and G

$$F(e_j) = e_j,\ F(e_{\bar{\jmath}}) = -e_{\bar{\jmath}},\ G(e_j, e_k) = G(e_{\bar{\jmath}}, e_{\bar{k}}) = 0,\ G(e_j, e_{\bar{k}}) = G_{j\bar{k}}. \tag{34}$$

For the canonical connection ∇ we get from (11) and (12)

$$\nabla_{e_j} e_k = \Gamma^i_{jk} e_i,\ \nabla_{e_j} e_{\bar{k}} = 0,\ \nabla_{e_{\bar{\jmath}}} e_k = 0,\ \nabla_{e_{\bar{\jmath}}} e_{\bar{k}} = \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} e_{\bar{\imath}}, \tag{35}$$

where

$$\Gamma^i_{jk} = G^{i\bar{\ell}} \partial_j G_{k\bar{\ell}},\ \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = G^{\bar{\imath}\ell} \partial_{\bar{\jmath}} G_{\bar{k}\ell}. \tag{36}$$

For the torsion T of ∇ we have

$$T(e_j, e_k) = T^i_{jk} e_i,\ T(e_j, e_{\bar{k}}) = 0,\ T(e_{\bar{\jmath}}, e_k) = 0,\ T(e_{\bar{\jmath}}, e_{\bar{k}}) = T^{\bar{\imath}}_{\bar{\jmath}\bar{k}} e_{\bar{\imath}}, \tag{37}$$

where

$$T^i_{jk} = G^{i\bar{\ell}}(\partial_j G_{k\bar{\ell}} - \partial_k G_{j\bar{\ell}}),\ T^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = G^{\bar{\imath}\ell}(\partial_{\bar{\jmath}} G_{\bar{k}\ell} - \partial_{\bar{k}} G_{\bar{\jmath}\ell}). \tag{38}$$

Finally, for the curvature R of ∇ we obtain

$$R_{e_j e_k} = 0,\ R_{e_{\bar{\jmath}} e_k} e_\ell = R^i_{\bar{\jmath}k\ell} e_i,\ R_{e_j e_{\bar{k}}} e_{\bar{\ell}} = R^{\bar{\imath}}_{j\bar{k}\bar{\ell}} e_{\bar{\imath}},\ R_{e_{\bar{\jmath}} e_{\bar{k}}} = 0, \tag{39}$$

where

$$R^i_{\bar{j}k\ell} = \partial_{\bar{j}} \Gamma^i_{k\ell}, \; R^{\bar{i}}_{j\bar{k}\bar{\ell}} = \partial_j \Gamma^{\bar{i}}_{\bar{k}\bar{\ell}}. \tag{40}$$

Particularly, in the case of a para-Kählerian structure, from $T = 0$ one obtains

$$\partial_j G_{k\bar{\ell}} = \partial_k G_{j\bar{\ell}}, \; \partial_{\bar{j}} G_{\bar{k}\ell} = \partial_{\bar{k}} G_{\bar{j}\ell}$$

and therefore, there exist some local functions A_ℓ and $A_{\bar{\ell}}$ on M, so that

$$G_{k\bar{\ell}} = \partial_k A_{\bar{\ell}}, \; G_{\bar{k}\ell} = \partial_{\bar{k}} A_\ell.$$

But G being symmetric, it follows $\partial_k A_{\bar{\ell}} = \partial_{\bar{k}} A_\ell$ and so, there exist a local function B on M so that $A_k = \partial_k B$, $A_{\bar{k}} = \partial_{\bar{k}} B$. Therefore in the para-Kählerian case, we have

$$G_{j\bar{k}} = \frac{\partial^2 B}{\partial x^j \partial x^{\bar{k}}} = B_{j\bar{k}}. \tag{41}$$

Because $G^{j\bar{k}} = B^{j\bar{k}}$, where $B^{j\bar{k}}$ are the elements of the matrix $[B_{j\bar{k}}]^{-1}$, we obtain

$$\Gamma^i_{jk} = \frac{\partial^3 B}{\partial x^j \partial x^k \partial x^{\bar{\ell}}} B^{i\bar{\ell}}, \; \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \frac{\partial^3 B}{\partial x^{\bar{j}} \partial x^{\bar{k}} \partial x^{\ell}} B^{\bar{i}\ell}, \tag{42}$$

and

$$\begin{aligned} R^i_{\bar{j}k\ell} &= \frac{\partial^4 B}{\partial x^{\bar{j}} \partial x^k \partial x^\ell \partial x^{\bar{m}}} B^{i\bar{m}} + \frac{\partial^3 B}{\partial x^k \partial x^\ell \partial x^{\bar{m}}} \frac{\partial B^{i\bar{m}}}{\partial x^{\bar{j}}} \\ R^{\bar{i}}_{j\bar{k}\bar{\ell}} &= \frac{\partial^4 B}{\partial x^j \partial x^{\bar{k}} \partial x^{\bar{\ell}} \partial x^m} B^{\bar{i}m} + \frac{\partial^3 B}{\partial x^{\bar{k}} \partial x^{\bar{\ell}} \partial x^m} \frac{\partial B^{\bar{i}m}}{\partial x^j} \end{aligned} \tag{43}$$

Finally for the principal component of the tensor field $\tilde{R}$, we get

$$\tilde{R}_{j\bar{k}\ell\bar{m}} = -\frac{\partial^4 B}{\partial x^j \partial x^{\bar{k}} \partial x^\ell \partial x^{\bar{m}}} + \frac{\partial^3 B}{\partial x^j \partial x^\ell \partial x^{\bar{p}}} \frac{\partial^3 B}{\partial x^{\bar{k}} \partial x^{\bar{m}} \partial x^q} B^{\bar{p}q}. \tag{44}$$

7. Example

Let N be a C^∞-manifold endowed with a linear connection D and $\pi : TN \to N$ its tangent bundle. To each local chart (U, φ) in $x \in N$, with $\varphi(x) = (x^i)$, we associate on the total space TN the chart $(\pi^{-1}(U), \phi)$ in $z = (x, y)$, with $\phi(z) = (x^i, y^i)$, where $y = y^i \frac{\partial}{\partial x^i}$. For a function $f \in \mathcal{F}(N)$, let $f^v = f \circ \pi$ its *vertical lift*. For $\omega \in \mathcal{T}_1(N)$ and $S \in \mathcal{T}^1_1(N)$, given locally by $\omega = \omega_i(x) dx^i$, $S = S^i_j(x) \frac{\partial}{\partial x^i} \otimes dx^j$, we set

$$\gamma(\omega_j dx^j)_z = \omega_j(x) y^j, \; \gamma\left(S^i_j \frac{\partial}{\partial x^i} \otimes dx^j\right)_z = y^j S^i_j(x) \frac{\partial}{\partial y^i}. \tag{45}$$

We remark that $\gamma(\omega) \in \mathcal{F}(TN)$ and $\gamma(S) \in \mathcal{T}^1(TN)$. It is easy to prove that for two vector fields $A, B \in \mathcal{T}^1(TN)$ one has $A = B$ if and only if $A(\gamma\omega) = B(\gamma\omega)$ for each $\omega \in \mathcal{T}_1(N)$. To a vector field $X \in \mathcal{T}^1(N)$, we shall associate the *vertical lift* $X^v \in \mathcal{T}^1(TN)$ and the *horizontal* *lift* $X^h \in \mathcal{T}^1(TN)$ with respect to a linear connection D on the base manifold N, characterised respectively by

$$X^v(\gamma\omega) = (\omega(X))^v,\ X^h(\gamma\omega) = \gamma(D_X\omega),\ \forall\omega \in \mathcal{T}_1(N). \tag{46}$$

We have the following useful formulas

$$X^v(f^v) = 0,\ X^h(f^v) = (X(f))^v, \tag{47}$$

and

$$[X^v, Y^v] = 0,\ [X^h, Y^v] = (D_X Y)^v,\ [X^h, Y^h] = [X, Y]^h - \gamma(\Re_{XY}) \tag{48}$$

where $\Re_{XY}$ is the curvature tensor field for D.

Setting

$$F(X^v) = X^v,\ F(X^h) = -X^h,\ \forall X \in \mathcal{T}^1(N), \tag{49}$$

we obtain an almost product structure F on TN with the eigendistributions $V_1 = VTN$, the vertical distribution of the fibration, and $V_2 = HTN$, the horizontal distribution of the connection D (see [3] and [5]). Let g be a (pseudo)-Riemannian metric on N and G the pseudo-Riemannian metric on TN given by

$$G(X^v, Y^v) = G(X^h, Y^h) = 0,\ G(X^v, Y^h) = G(X^h, Y^v) = (g(X, Y))^v. \tag{50}$$

It is easy to check that the pair (F, G) satisfies the compatibility conditions (6) and therefore we have

Proposition 7.1. *The almost product structure F, associated to a linear connection D by* (49), *and the pseudo-Riemannian metric G, associated to D and the (pseudo)-Riemannian metric g by* (50), *determine an aph-structure (F, G) on the total space TN.*

In the following, we suppose that the connection D is a metric one for g, i.e., $Dg = 0$. From the formulas (11) and (12), taking account of (47), (48), (49), we obtain for the canonical connection ∇ associated to the aph-structure (F, G),

$$\nabla_{X^v} Y^v = 0,\ \nabla_{X^h} Y^v = (D_X Y)^v,\ \nabla_{X^v} Y^h = 0,\ \nabla_{X^h} Y^h = (D_X Y)^h. \tag{51}$$

So, we have

Proposition 7.2. *The canonical connection ∇, associated to the aph-structure (F, G) given by* (49) *and* (50) *on the total space TN, is the horizontal lift, in the sense of K. Yano and S. Ishihara* [9], *of the connection D on the base manifold N.*

For the torsion and curvature tensor fields T and R of the canonical connection ∇ we obtain

$$\begin{aligned} &T(X^v, Y^v) = T(X^v, Y^h) = 0,\ T(X^h, Y^h) = (t(X, Y))^h + \gamma(\Re_{XY}), \\ &R_{X^v Y^v} = R_{X^v Y^h} = 0,\ R_{X^h Y^h} Z^v = (\Re_{XY} Z)^v,\ R_{X^h Y^h} Z^h = (\Re_{XY} Z)^h, \end{aligned} \tag{52}$$

where t and $\Re_{XY}$ are the torsion and the curvature tensors for D. It follows

$$\overset{1}{T} = 0,\ \overset{2}{T}(X^h, Y^h) = (t(X, Y))^h,\ \overset{1}{S} = 0,\ \overset{2}{S}(X^h, Y^h) = \gamma(\Re_{XY}). \tag{53}$$

For the Nijenhuis tensor of the apc-strucure F we get

$$N(X^v, Y^v) = N(X^v, Y^h) = 0,\ N(X^h, Y^h) = -4\gamma(\Re_{XY}). \tag{54}$$

Finally, for the exterior derivative of the fundamental 2-form Ω, we find

$$\begin{array}{rcl} d\Omega(X^v, Y^v, Z^v) & = & d\Omega(X^v, Y^v, Z^h) = 0 \\ 3d\Omega(X^h, Y^h, Z^v) & = & -(g(t(X,Y), Z))^v, \\ 3d\Omega(X^h, Y^h, Z^h) & = & \gamma(\sum_{XYZ} i_X g \circ \Re_{YZ}). \end{array} \tag{55}$$

From the formulas (53), (54), (55) it results

Proposition 7.3. *The aph-structure (F, G) on the total space TN, associated to a (pseudo)-Riemannian metric g and a metric connection D on the base manifols N, by the relations* (49), (50), *is generally* 1*-para-Kählerian. It is almost para-Kählerian, para-Hermitian or para-Kählerian respectively if and only if the connection D is torsionless, has vanishing curvature or is both torsionless and with vanishing curvature.*

Final remark. Let (F, G) be an aph-structure on a manifold M, ∇ the canonical connection, T its torsion and $\Psi$the tensor field given by $\Psi(X, Y, Z) = G((X, T(Y, Z))$. One can prove that Ψ has the same symmetries as the tensor field $\Phi = \tilde{\nabla}\Omega$ (where $\tilde{\nabla}$ is the Levi-Civita connection of G), considered independently in [1] and [6] for to obtain the classification of the aph-manifolds. Hence one can obtain classification of the aph-manifolds, given in terms of Ψ, which may coincides with the classification based on Φ, but with different characterizations.

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